

Needham, Chapter 11, Exercise 2, p. 505.

G. Palmer, April 22, 2018, 8:10 am PDT

2 For each of the following vector fields X , calculate $F[X, C]$ and $W[X, C]$ for the given loop C , then check your answers by substituting the results of the previous question into (7) and (8).

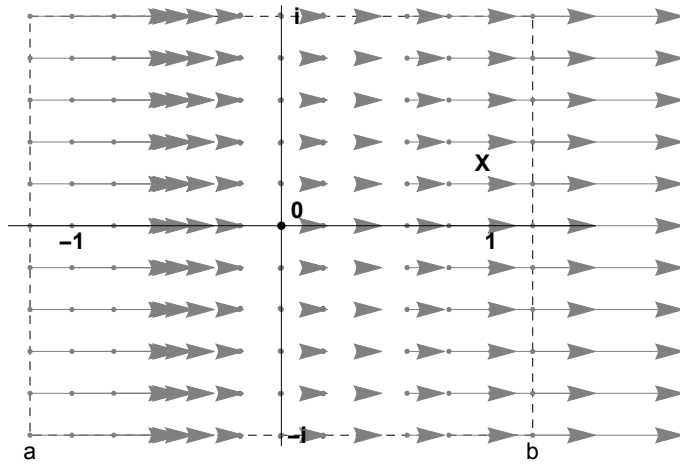


Figure 1. $X = x^2$

(i) $X = x^2$, and C is the edge of the rectangle $a \leq x \leq b$, $-1 \leq y \leq 1$, traversed counter clockwise.

Flux

We use the approach described on p. 485, ¶ 1 and write

$$F[X, C] = (\text{fluid out}) - (\text{fluid in})$$

There is no flux across lines parallel to the x axis. All the flux crosses the vertical sides of C and flows eastward. The length of the vertical sides of C is 2. The speed of X is $|X| = x^2$. Then the flux is calculated at $x = a$ and $x = b$ as the speed times the arc length. The flux at a is therefore $2a^2$, and the flux at b is $2b^2$. Then

$$F[X, C] = (2b^2 - 2a^2) = 2(b^2 - a^2)$$

In exercise 1 (iii) we calculated that the divergence $\nabla \cdot X = 2x$.

By (7),

$$F[X, C] = \iint_R [\nabla \cdot X] dA$$

$$\begin{aligned}
&= \int_{-1}^1 \int_a^b 2x \, dx \, dy \\
&= \int_{-1}^1 [x^2]_a^b \, dy \\
&= \int_{-1}^1 (b^2 - a^2) \, dy \\
&= (b^2 - a^2)[y]_{-1}^1 \\
&= 2(b^2 - a^2)
\end{aligned}$$

This verifies the first result.

Work

Let E, W, N, and S designate the four sides of C. Traversing C in the clockwise direction, and noting that the curl on the E and W sides is zero, because the sides are orthogonal to the streamlines,

$$\begin{aligned}
W[X, C] &= -W[X, S] + 0 + W[X, N] + 0 \\
&= W[X, N] - W[X, S]
\end{aligned}$$

It is easy to see from figure 1 that $W[X, N] = W[X, S]$, so $W[X, C] = 0$.

In Exercise 1 (iii) we calculated $\nabla \times X = 0$. By (8),

$$\begin{aligned}
W[X, C] &= \iint_R [\nabla \times X] \, dA \\
&= \iint_R [0] \, dA = 0
\end{aligned}$$

This verifies the first result.

(ii) $X = i(1/r^2)e^{j\theta}$, and C is the edge of the region $a \leq r \leq b$, $0 \leq \theta \leq \pi$, traversed counterclockwise.

Flux

From Exercise 1 (v), we know that the streamlines are circular arcs (Ex. 1, Figures 7, 8). The tangents to the streamlines are normal to the rays, so there is no flux across the arcs, but there is flux across the rays. We are interested in the net flux across the ray segments (b-a) at $\theta = 0$ and $\theta = \pi$.

The speed is $|X| = 1/r^2$. One can see from the figures that the flux at every point on (b-a) is equal at

$\theta = 0$ and $\theta = \pi$. Therefore, the net flux is 0.

Check with (8). We know from Exercise 1 (v) that the divergence is zero. Therefore,

$$\begin{aligned} F[X, C] &= \iint_C [\nabla \cdot X] \, dA \\ &= \iint_C 0 \, dA = 0 \end{aligned}$$

The first result is verified by (8).

Work

The inner radius is a and the outer is b , so the smaller arc has length πa and the larger arc has length πb . Since X is tangent to the streamlines, which are segments of concentric circles, there is no curl on the rays, which are orthogonal to X . The net work is the sum of the work on the arcs. The work along the inner arc is the speed $(1/a^2)$ times arc length (πa) , but with a minus sign because it opposes the counterclockwise direction of traversal: $-(1/a^2)(\pi a)$. The work along the outer arc is $(1/b^2)(\pi b)$. The net work is

$$W[F, C] = \pi/b - \pi/a$$

Check with (8). In Exercise 1 (v), the curl was calculated to be $-1/r^3$.

$$W[F, C] = \iint_C [\nabla \times X] \, dA$$

$$dA = r \, dr \, d\theta$$

To find dA , plot a circle on r , a second circle on $r + dr$, and two rays with angle θ between them. The arc length is $r \, d\theta$. Then $dA = dr (r \, d\theta)$.

$$\begin{aligned} W[F, C] &= - \int_0^\pi \int_a^b r^{-3} r \, dr \, d\theta \\ &= - \int_0^\pi \int_a^b r^{-2} \, dr \, d\theta \\ &= - \int_0^\pi [1/r]_a^b \, d\theta \\ &= - \int_0^\pi [1/b - 1/a] \, d\theta \\ &= \frac{\pi}{b} - \frac{\pi}{a} \end{aligned}$$

This calculation supports the original. But calculating using $\text{curl} \times (\text{area of the annulus segment } a \leq r \leq b)$ produces a different answer.

$$\text{area} = (\theta/2\pi) \pi r^2 = (1/2) \pi r^2$$

$$\text{area under outer arc} = (1/2) \pi b^2$$

$$\text{area under inner arc} = (1/2) \pi a^2$$

$$\text{area annulus} = (1/2) \pi b^2 - (1/2) \pi a^2$$

$$\text{work} = \text{curl} \times \text{area annulus} = \left(-1/r^3\right) (\text{area})$$

How does one apply this? The following produces a wrong answer.

$$\text{work on outer arc} = \left(-1/b^3\right) (1/2) \pi b^2 = -\pi / (2b)$$

$$\text{work on inner arc} = -\left(-1/a^3\right) (1/2) \pi a^2 = \pi / (2a)$$

$$\text{net work} = \pi / (2a) - \pi / (2b)$$

$$\text{net work} = \frac{\pi}{2a} - \frac{\pi}{2b}$$

This is only half right!