

Needham, Chapter 11, Exercise 17, p. 507.
 G. Palmer, October 24, 2018, 11:45 am PDT

17 Show that the streamlines [15b] and the equipotentials of a quadrupole are lemniscates (see [9], p. 62).

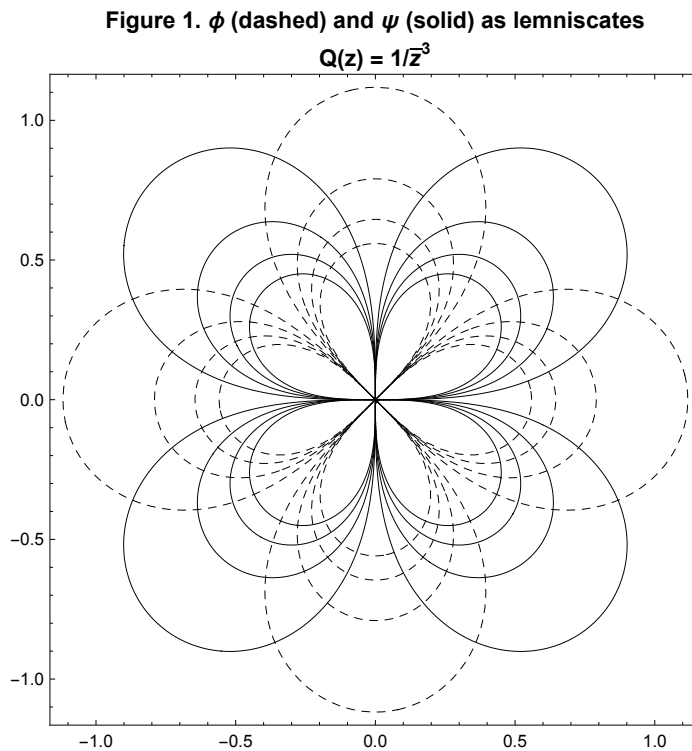


Figure 1 shows the streamlines in solid lines and the potentials in dashed lines. The streamlines and the potentials each form a quadrupole, but to plot a streamline quadrupole, one must use both negative and positive values of ψ . Both the streamlines and the potentials are rotated $\pi/4$ radians from the normal orientations of cassinian curves and quadrupoles.

Let $\bar{H}(z) = Q(z) = k/\bar{z}^3$ (p. 492).

$$H = k/\bar{z}^3$$

$$\Omega = \int_K (k/\bar{z}^3) dz = -k/2 \bar{z}^2$$

$$= -(k/2) \bar{z}^2 / |z|^4$$

$$= -(k/2) (x^2 - y^2 - 2ixy) / (x^2 + y^2)^2$$

Take real and imaginary parts to obtain ϕ and ψ :

$$\phi = (k/2) (y^2 - x^2) / (x^2 + y^2)^2$$

We note that $\phi = 0$ when $|y| = |x|$, except at the origin, where it is undefined.

$$\psi = kxy / (x^2 + y^2)^2$$

$\psi = 0$ when $x = 0$ or $y = 0$, except at the origin, where it is undefined. These are the equations plotted with ContourPlot[] in Figure 1. A negative version of ψ was necessary to account for the 2nd and 3rd quadrants, where the product xy is negative.

Using algebra, we must show that the equations for ϕ and ψ are equivalent to $r^2 = 2 \cos 2\theta$. Let $x = r \cos \theta$, $y = r \sin \theta$. Then

$$\phi = (k/2) r^2 (\sin^2 \theta - \cos^2 \theta) / r^4 (\cos^2 \theta + \sin^2 \theta)^2$$

$$= (k/2r^2) \cos 2\theta$$

$$\psi = kr^2 \cos \theta \sin \theta / r^4 (\cos^2 \theta + \sin^2 \theta)^2$$

$$= (k/2r^2) \sin 2\theta$$

Then if we let $k = 4$, and $r = 1$

$$\phi = 2 \cos 2\theta$$

which is equivalent to (2) on p. 62, where $\phi = r^2 = \text{const.}$ The corresponding equation for ψ is

$$\psi = 2 \sin 2\theta$$

This also plots lemniscates but with a phase difference of $\pi/4$ from ϕ .