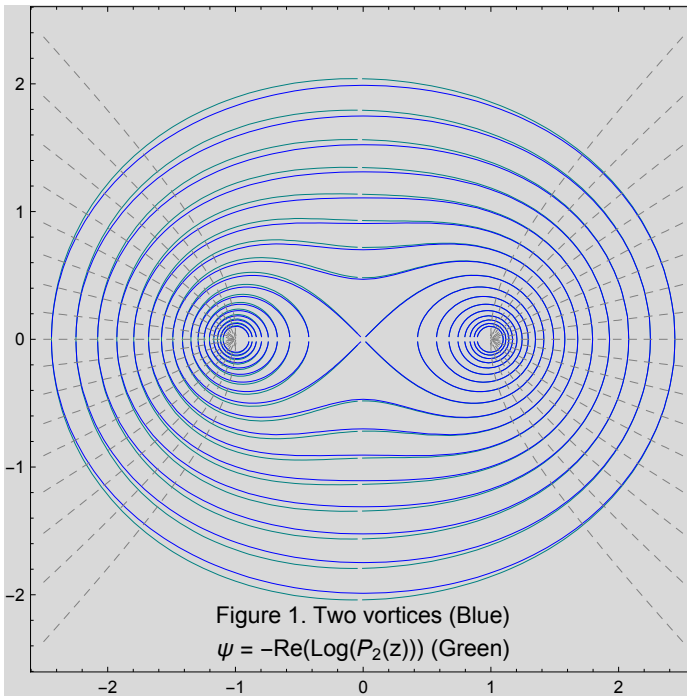


Needham, Chapter 11, Exercise 16, p. 507.
 G. Palmer, October 23, 2018, 2:45 pm PDT

16 Show that the streamlines produced by two vortices of equal strength are Cassinian curves (figure 8[b], p. 61) whose foci are the locations of the two vortices. [Note that your reasoning immediately generalizes: Cassinian curves with n foci are the streamlines of n equal vortices placed at the foci.]



We guess from the equation for a vortex (p. 488) that the velocity at z of the superimposed vortices is a Pólya vector $\overline{H}$. Let a_1 and a_2 be the foci of the two vortices. We need ψ .

$$\overline{H}(z) = \frac{i}{\bar{z}-a_1} + \frac{i}{\bar{z}-a_2}$$

$$H(z) = \frac{-i}{z-a_1} + \frac{-i}{z-a_2}$$

$$\Omega(z) = \int_K H(z) dz = -i [\text{Log}(z-a_1) + \text{Log}(z-a_2)]$$

$$= -i \text{Log}[(z-a_1)(z-a_2)]$$

$$= -i [\ln|(z-a_1)(z-a_2)| + i \arg((z-a_1)(z-a_2))]$$

$$= -i \ln|(z-a_1)(z-a_2)| + \arg((z-a_1)(z-a_2))$$

$$\psi = \text{Im}[\Omega] = -\ln|(z-a_1)(z-a_2)|$$

(1)

The equation is plotted as the blue family of curves in Figure 1. We need to show that this is a genuine Cassinian curve defined algebraically. In Chapter 2, p. 63, we find that Cassinian curves in the z -plane are the preimages of origin-centred circles $|w| = \text{const.}$, and as such *the circles* can be expressed as polynomials whose roots are the Cassinian foci:

$$z \mapsto w = P_n(z) = (z - a_1)(z - a_2) \cdots (z - a_n)$$

In this case,

$$P_2(z) = (z - a_1)(z - a_2) \quad (2)$$

How can (2), the image circles, be transformed into their Cassinian preimages (1)? Find the complex log of both sides.

$$\begin{aligned} \text{Log}(P_2(z)) &= \text{Log}((z - a_1)(z - a_2)) \\ &= \ln|(z - a_1)(z - a_2)| + i \arg((z - a_1)(z - a_2)) \end{aligned}$$

Multiply through by $-i$ to obtain $\Omega(z)$.

$$-i \text{Log}(P_2(z)) = -i \ln|(z - a_1)(z - a_2)| + \arg((z - a_1)(z - a_2)) = \Omega(z)$$

$$\psi = -\ln|(z - a_1)(z - a_2)| = \text{Im}(-i \text{Log}(P_2(z))) = -\text{Re}(\text{Log}(P_2(z)))$$

This shows that the streamlines of the transposed vortices are the Cassinian preimages $-\text{Re}(\text{Log}(P_2(z)))$ of $P_2(z)$.

It is trivial to show that the contour plot of $\text{Re}(\text{Log}(P_n(z)))$ is exactly the same as the contour plot of the transposed vortices. For the sake of illustration, Figure 1 plots the streamlines of the transposed vortices in blue and the Cassinian preimages in green. The green curves are offset slightly to the west to avoid covering the blue curves. The dashed lines are lines of potential calculated on $\text{Im}(-\text{Log}(P_n(z)))$.

Following Vasco's much more direct path, we can rewrite ψ as

$$\psi = -\ln(|(z - a_1)| |(z - a_2)|)$$

Then

$$e^{-\psi} = |(z - a_1)| |(z - a_2)|$$

Since the stream function ψ is a level curve (constant), $e^{-\psi}$ is also constant and this equation meets the criterion set forth in (1), p. 61, that a Cassinian curve is the result when the product of the distances from z to the foci is constant. Therefore, this equation derived from two vortices describes a Cassinian curve with foci at a_1 and a_2 .

