

Needham, Chapter 11, Exercise 15, p. 507.
G. Palmer, October 23, 2018 2:50 pm

15 If two sources are located at opposite corners of a square, and two sinks are located at the other two corners, and all four are of equal strength, then show that the circle through these four points is a streamline. Check this by getting the computer to draw the complete flow.

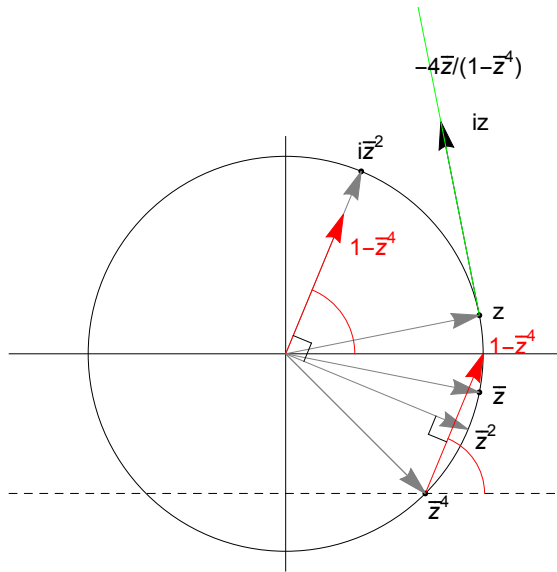


Figure 1. Geometry of proof
that $\arg(1 - \bar{z}^4) = 2 \arg(\bar{z}) + \pi/2$

The proposed construction has symmetry about both axes and the axes and the circle has symmetry about the origin (Figure 2). This means a very abstract construction with a unit circle centred at the origin should be adequate for the proof whatever the actual centre or degree of rotation about it. I don't know in what respect the "the complete flow" would be complete, but the streamlines are plotted in Figure 2.

Algebraic/Geometric Proof (Figure 1)

Centre a square at the origin with the four corners at $\{A, B, -A, -B\}$, where $B = Ae^{i\pi/2}$. Place the sources at $\{A, -A\}$ and the sinks at $\{B, -B\}$, all with strength $S = 2\pi$. Then

$$\bar{H} = \frac{1}{\bar{z}-A} - \frac{1}{\bar{z}-B} + \frac{1}{\bar{z}+A} - \frac{1}{\bar{z}+B} \quad (\text{the Pólya field})$$

$$= -\frac{2(A^2 z - B^2 z)^*}{(A^* - z^*)(A+z)^*(z^* - B^*)(B+z)^*} \quad (\text{This calculation done by mathematica. } X^* = \bar{X})$$

$$\begin{aligned}
 H &= \frac{1}{z-A} - \frac{1}{z-B} + \frac{1}{z+A} - \frac{1}{z+B} \\
 &= -\frac{2(A^2 z - B^2 z)}{(A-z)(A+z)(z-B)(B+z)}
 \end{aligned}$$

We need algebraic or geometric proof that the circle is a streamline. If $\overline{H}(z)$ touches C (is tangent to C), then it is perpendicular to a ray from the origin that intersects C at z , so $\overline{H} \cdot z = 0$.

$$\begin{aligned}
 \overline{H} \cdot z &= -\frac{2(A^2 z - B^2 z)^*}{(A^* - z^*)(A+z)^*(z^* - B^*)(B+z)^*} \cdot z \\
 &= -\frac{2(A^2 z - B^2 z)^*}{(-A^2 B^2 + z^2 B^2 + A^2 z^2) - (z^*)^4} \cdot z
 \end{aligned}$$

We also know that $B = Ae^{i\pi/2}$, so $B^2 = A^2 e^{i\pi} = -A^2$

$$\begin{aligned}
 \overline{H} \cdot z &= -\frac{2(A^2 z + A^2 z)^*}{(A^4 - z^2 A^2 + A^2 z^2) - (z^*)^4} \cdot z \\
 &= -\frac{4(A^2 z)^*}{(A^4)^* - (z^*)^4} \cdot z
 \end{aligned}$$

This proved difficult to reduce to an expression with x and y that could be put in complex vector format. Since the construction has rotational symmetry about the origin, let $A = 1$ and try to show that $\arg\left(-\frac{4z^*}{1-(z^*)^4}\right) = \arg(i z) + n\pi$, $n = \pm 1, \pm 2, \dots$. We can show this to be the case empirically. With algebra, we have

$$\arg\left(-\frac{4\bar{z}}{1-(\bar{z})^4}\right) = \arg(\bar{z}) - \arg(1 - (\bar{z})^4) + \pi$$

and

$$\arg(iz) = \arg[z] + \pi/2 = -\arg(\bar{z}) + \pi/2$$

Then the right hand sides should be equal:

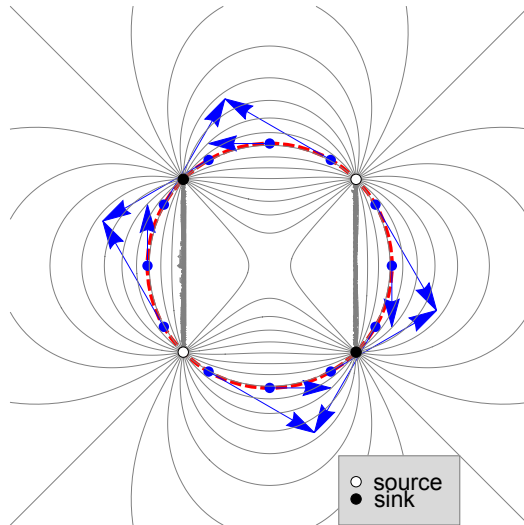
$$\arg(\bar{z}) - \arg(1 - (\bar{z})^4) + \pi = -\arg(\bar{z}) + \pi/2$$

$$\arg(1 - (\bar{z})^4) = \arg(\bar{z}^2) + \pi/2$$

Now it is clear that we want to show that $(1 - (\bar{z})^4)$ is parallel to $i\bar{z}^2$. We can do this with geometry by showing that both vectors are orthogonal to a third line segment, $\bar{z}^2$. The vector $i\bar{z}^2 \perp \bar{z}^2$ by definition. $(1 - (\bar{z})^4) \perp \bar{z}^2$ because it is bisected by the spoke $\bar{z}^2$ of C.

We have shown that $\overline{H}$ is orthogonal to z and is therefore tangent to all points of the circle except the sources and sinks themselves, which are critical points. So the four segments of the circle must be streamlines.

Figure 2. Sources and sinks at corners of square



Plotting the flow and calculating ϕ and ψ by integrating $\overline{H}$

$$\Omega = \int H dz = \log(z-A) - \log(z-B) + \log(z+A) - \log(z+B)$$

$$\log(z) = \ln|z| + i \arg(z)$$

$$\arg(z) = \arctan(\text{Im}(z)/\text{Re}(z))$$

We are given that $H = \Omega'$. Then, ϕ and ψ must be the real and imaginary parts of $\Omega(z) = \int H dz$.

$$H = \frac{1}{z-A} - \frac{1}{z-B} + \frac{1}{z+A} - \frac{1}{z+B}$$

$$\Omega = \text{Log}(z-A) - \text{Log}(z-B) + \text{Log}(z+A) - \text{Log}(z+B)$$

$$= \ln|z-A| + i\arg(z-A) - (\ln|z-B| + i\arg(z-B)) \dots$$

$$= \ln\sqrt{(x-A)^2 + y^2} + i \arctan\left(\frac{\text{Im}(z-A)}{\text{Re}(z-A)}\right)$$

$$- (\ln\sqrt{(x-B)^2 + y^2} + i \arctan\left(\frac{\text{Im}(z-B)}{\text{Re}(z-B)}\right))$$

$$+ \ln\sqrt{(x+A)^2 + y^2} + i \arctan\left(\frac{\text{Im}(z+A)}{\text{Re}(z+A)}\right)$$

$$\begin{aligned}
& - (\ln \sqrt{(x+B)^2 + y^2} + i \arctan(\frac{\text{Im}(z+B)}{\text{Re}(z+B)})) \\
& = \ln \left(\sqrt{(x-A)^2 + y^2} \ln \sqrt{(x-B)^2 + y^2} \ln \sqrt{(x+A)^2 + y^2} \ln \sqrt{(x+B)^2 + y^2} \right) + i \\
& [\arctan(\frac{\text{Im}(z-A)}{\text{Re}(z-A)}) - \arctan(\frac{\text{Im}(z-B)}{\text{Re}(z-B)}) + \arctan(\frac{\text{Im}(z+A)}{\text{Re}(z+A)}) - \arctan(\frac{\text{Im}(z+B)}{\text{Re}(z+B)})]
\end{aligned}$$

We want the real and imaginary parts of Ω .

$$\begin{aligned}
\phi &= \ln \left(\sqrt{(x-A)^2 + y^2} \sqrt{(x-B)^2 + y^2} \sqrt{(x+A)^2 + y^2} \sqrt{(x+B)^2 + y^2} \right) \\
\psi &= \arctan(\frac{\text{Im}(z-A)}{\text{Re}(z-A)}) - \arctan(\frac{\text{Im}(z-B)}{\text{Re}(z-B)}) + \arctan(\frac{\text{Im}(z+A)}{\text{Re}(z+A)}) - \arctan(\frac{\text{Im}(z+B)}{\text{Re}(z+B)})
\end{aligned}$$

ψ is used to plot the streamlines in Figure 2. One can see by inspection that the red circle fits into the pattern of streamlines and that Pólya vector field is tangent to the red circle.

Computer demonstration

We are to show that the red, dashed circle that passes through the four points is a streamline. If it is, then the field vectors should be tangent to the circle C with center q . Since the sources and sinks are Pólya fields, $\overline{H}(z)$ should be tangent to C at any z on C .

$$\overline{H} = \overline{\Omega}' = - \frac{2(A^2 z - B^2 z)^*}{(A^* - z^*)(A+z)^*(z^* - B^*)(B+z)^*}$$

The circle that passes through the four points can be described as $|z| = 1$. We can also write this circle as

$$z = e^{i\theta}, \text{ where } \theta = [0..2\pi]$$

By plotting $\overline{H}(z)$ at $z = e^{i\theta}$, where $\theta = n\pi/6$ and $n = 1, 2, 3, \dots, 12$, we determine by inspection that all the blue vectors $\overline{H}(z)$ are tangent to the circle, as we hoped to show. This is visual evidence that the circle is a streamline, except that it consists of four segments; there are no tangents at the sources and sinks. Streamlines are open sets that begin and end at the critical points without including them.