

Needham, Chapter 11, Exercise 14, p. 507.
 G. Palmer, October 23, 2018, 2:45 pm PDT

14 Suppose that a doublet consisting of a source of strength 2π at $z = 1$ and a sink of equal strength at $z = -1$ is inserted into the uniform flow with real, positive velocity v . Locate the “stagnation point” (singular points of zero velocity) of the net flow, and describe (perhaps with the aid of a computer animation) how they move as v varies from 0 to 3.

P. 488 provides equations for sources and sinks of strength 2π . Adapting them for this problem, one can write

$$\text{source: } \bar{H}_{\text{src}} = 1/(\bar{z}-1) \implies H_{\text{src}} = 1/(z-1)$$

$$\text{sink: } \bar{H}_{\text{snk}} = -1/(\bar{z}+1) \implies H_{\text{snk}} = -1/(z+1)$$

$$H = \Omega' = v + 1/(z-1) - 1/(z+1)$$

At the stagnation point, $\bar{H} = H = 0$.

$$\bar{H} = \Omega' = v + 1/(\bar{z}-1) - 1/(\bar{z}+1) = 0$$

$$v(\bar{z}-1)(\bar{z}+1) + (\bar{z}+1) - (\bar{z}-1) = 0$$

$$v(\bar{z}-1)(\bar{z}+1) = -2$$

$$v\bar{z}^2 = v - 2$$

$$\bar{z}^2 = (v-2)/v$$

$$z = \pm \sqrt{(v-2)/v}, \quad \text{because } (v-2)/v \text{ is real}$$

The red points in Figures 1[a,b,c] represent this solution to the points of stagnation. As velocity increases, they appear to merge at the centre and then approach the source and sink along the real line. I first surmised that the points of stagnation might occur in areas where the streamlines are farthest apart or where there are large blank spaces with no streamlines or lines of potential. This appears to be correct, but incomplete and imprecise. Then Vasco pointed out that I could calculate the points by setting the velocity to 0. I surmised that the velocity must be the Pólya vector $\bar{H}$. It might help to plot the Pólya vectors on a grid. Vasco also analyzed the positions of the points of stagnation for various values of v . I attempt to reproduce his analysis.

We will examine z when $v = 0$, $0 < v \leq 2$, $2 < v \leq 3$.

When $v = 0$, z is undefined.

When $0 < v \leq 2$, the value under the root symbol is negative, so $z = \pm i \sqrt{|(v-2)/v|}$. This means z is on the y axis. As v approaches 0 from above, $z = \pm i \infty$. When v goes to 2, $z = 0$. At $v = 1$, $z = \pm i$,

so it lies on the unit circle.

When $2 < v \leq 3$, the value under the root symbol is positive, so $z = \pm \sqrt{|(v-2)/v|}$. When v is very near 2, z approaches 0. When v reaches 3, $z = \pm \sqrt{1/3}$. z travels away from the origin and towards $\pm \sqrt{1/3}$.

The remainder of this answer deals with plotting the streamlines. We do not introduce a specific contour.

$$\begin{aligned}
 \Omega &= \int H(z) dz = \int (v + 1/(z-1) - 1/(z+1)) dz \\
 &= vz + \text{Log}(z-1) - \text{Log}(z+1) + \text{const.} = W + iF \\
 &= vz + \ln|z-1| + i(\arg(z-1) + n2\pi) - \ln|z+1| - i(\arg(z+1) + n2\pi) \\
 &\quad (n = 0, 1, 2, \dots, \text{ but we use just the principle value of Log with } n = 0. \\
 &\quad \text{The constants cancel.}) \\
 &= v(x+iy) + \ln|x+iy-1| + i(\arg(x+iy-1) + n2\pi) - \ln|x+iy+1| - i\arg(x+iy+1) \\
 &= v(x+iy) + \ln(\sqrt{(x-1)^2 + y^2}) - \ln(\sqrt{(x+1)^2 + y^2}) + i(\arg(x+iy-1) - \arg(x+iy+1)) \\
 \psi &= vy + \arg(x+iy-1) - \arg(x+iy+1) \\
 &= vy + \arctan\left[\frac{y}{x-1}\right] - \arctan\left[\frac{y}{x+1}\right]
 \end{aligned}$$

or

$$vy + \arg\left(\frac{x^2+y^2+2iy-1}{x^2+2x+y^2+1}\right) = vy + \arctan\left(\frac{\text{Im}(x^2+y^2+2iy-1)}{\text{Re}(x^2+y^2+2iy-1)}\right) = vy + \arctan(2y/(x^2+y^2-1))$$

The second equation gives a better plot, and the fact that the dashed potential lines of ϕ are orthogonal to the lines of flow is encouraging. Notice that the arctan goes to $\pm \pi/2$ on the unit circle, so $\psi = vy \pm \pi/2$.

$$\phi = vx + \ln \sqrt{\frac{x^2+y^2-2x+1}{x^2+y^2+2x+1}}$$

An animated GIF file accompanies this document. The unit circle was plotted by the *Mathematica* ContourPlot[] function. We don't know why, but perhaps it has to do with the accidental plotting of connecting lines between the ends of lines of flow due to some feature of the Mesh option. The plots should be symmetrical about the real line. *Mathematica* renders them asymmetrical, but they are accurate.

Figure 1[a].
 $\psi = vy + \arctan(2y/(x^2+y^2-1))$
 $v = 1, k = 0.1$
 $-(1+\pi/2) \leq \psi \leq (1+\pi/2)$

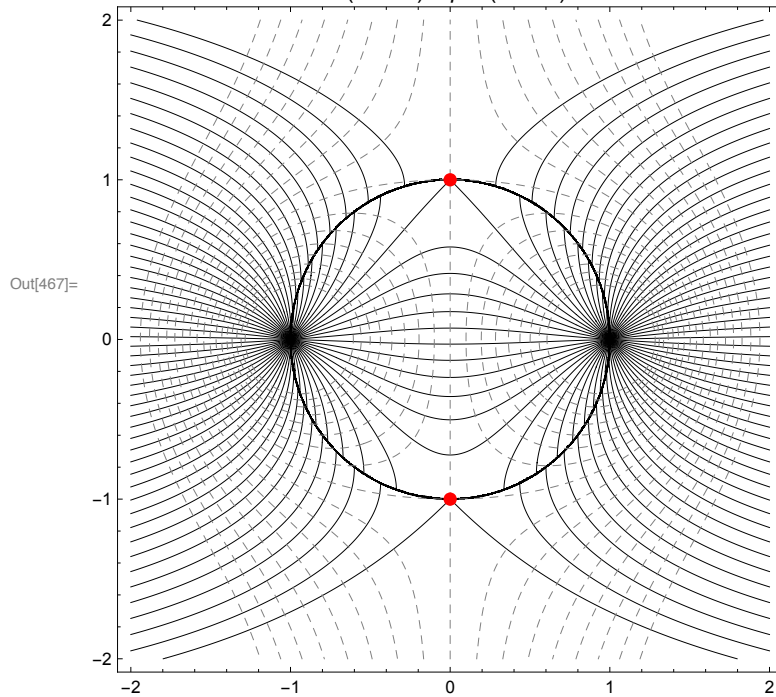


Figure 1[b].
 $\psi = vy + \arctan(2y/(x^2+y^2-1))$
 $v = 2, k = 0.1$
 $-(1+\pi/2) \leq \psi \leq (1+\pi/2)$

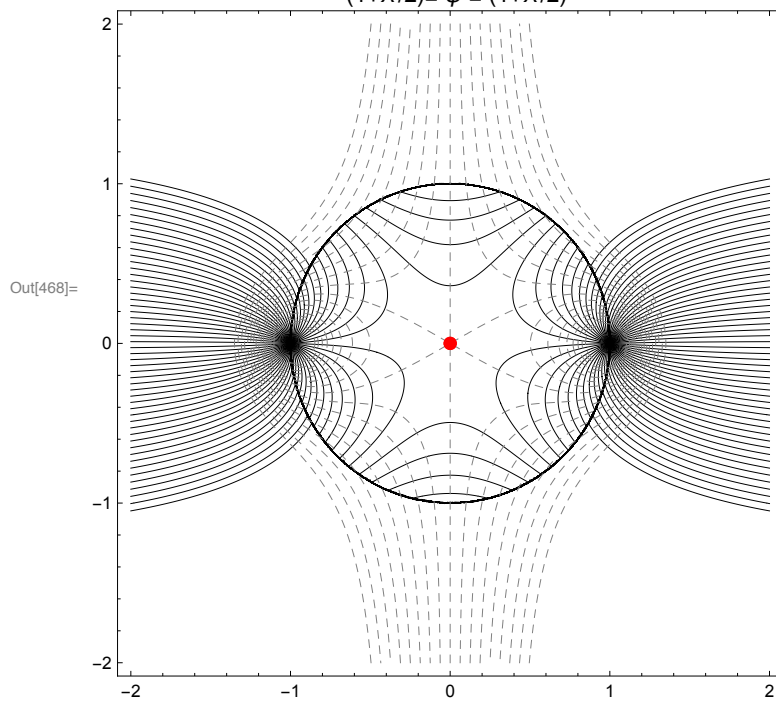


Figure 1[c].
 $\psi = vy + \arctan(2y/(x^2+y^2-1))$
 $v = 3, k = 0.1$
 $-(1+\pi/2) \leq \psi \leq (1+\pi/2)$

