

Needham, Chapter 11, Exercise 12, p. 507.
G. Palmer, September 2, 2018, 8 pm PDT

12 Consider the flow with complex potential

$$\Omega(z) = \frac{1}{2} \left[\frac{e^z + 1}{e^z - 1} \right].$$

Use a computer to draw the streamlines, and verify mathematically that the picture may be interpreted as the flow that results when the dipole with complex potential $\Omega(z) = (1/z)$ is confined to the channel $-\pi \leq \text{Im}(z) \leq \pi$.

$\Omega_A(z) = 1/z$, **Figure 1 (Blue) a dipole**

$$\Omega_A(z) = 1/z = 1/(x + iy)$$

$$= (x - iy) / (x^2 + y^2)$$

$$\psi = \text{Im}(\Omega) = -y / (x^2 + y^2)$$

$\Omega_B(z) = \frac{1}{2} \left[\frac{e^z + 1}{e^z - 1} \right]$ **Figure 1 (RedBrown), a series of dipole-like streamlines with period 2π**

$$w(z) = \Omega_B(z) = \frac{1}{2} \left[\frac{e^z + 1}{e^z - 1} \right]$$

$$= \frac{1}{2} \left[\frac{e^z + 1}{e^z - 1} \cdot \frac{\overline{e^z - 1}}{\overline{e^z - 1}} \right] = \frac{1}{2} \frac{e^z \bar{e}^z - e^z + \bar{e}^z - 1}{e^z \bar{e}^z - e^z - \bar{e}^z + 1}$$

$$= \frac{1}{2} \frac{e^{2x} - e^x(e^{iy} - e^{-iy}) - 1}{e^{2x} - e^x(e^{iy} + e^{-iy}) + 1} = \frac{1}{2} \frac{e^{2x} - 2i e^x \sin(y) - 1}{e^{2x} - 2e^x \cos(y) + 1}$$

$$\psi = \text{Im}[\Omega] = \frac{-e^x \sin(y)}{e^{2x} - 2e^x \cos(y) + 1}$$

Each contour in Figure 1 (whether blue or red-brown) represents $\psi = (n/k) = (n \times .05)$, where $n = \pm 1, \pm 2, \pm 3, \dots$. It is necessary to use a small value of k to show that the streamlines of $\Omega_B(z)$ approach their boundaries at $\pm n i \pi$, where $n = 1, 2, 3, \dots$. Large values of k reduce smearing at the pole. Smaller values of k create more data points, but one can compensate by reducing the range of k . See the figure heading for actual values. The *Mathematica* statement which plots a single contour is

ContourPlot[f(x, y) = ψ , {x, min, max}, {y, min, max}].

From the problem statement, one might expect true circles for the streamlines of $\Omega_B(z)$, but one might also interpret flattened circles as flattened or squashed streamlines of dipoles. Since the flux

is confined above and below, a streamline must bulge east or west in order to conserve diameter and flux.

The analytic function e^z is a component of Ω_B . The component y of $z = x + iy$ functions as the arg of $e^z = e^x e^{iy}$, so e^z has a period of 2π . Since $e^z = e^x e^{iy} = e^x (\cos y + i \sin y)$, the vertical component of z goes to zero at $\pm\pi$. Thus, it makes sense to confine the dipole to a channel of $-\pi \leq \text{Im}[z] \leq \pi$. We can see by inspection of Figure 1 that one of the flattened streamline clusters of $\Omega_B(z)$ is approximately confined to the channel. One could think of it as the confinement of a dipole $1/z$ to the channel.

We have to show mathematically what happens to streamlines of the blue dipole if $|\text{Im}(z)|$ can never exceed π for $\Omega(z) = 1/z$. The streamlines of $\Omega(z)$ are $\Omega^{-1}(z)$, which is just $1/z$. This has an imaginary part of $\frac{-y}{x^2+y^2}$, which in this case equals the value of ψ . We want that $|y|$ not exceed π . x has no such constraint. Hence, if x is large, the plot bulges to the east and west because y cannot increase beyond π or decrease beyond $-\pi$. As the velocity of the flow increases at the top and bottom, the k-tubes narrow, giving the visual appearance of a squashed object. If we set ψ to any value, where

$$\psi(x,y) = \frac{-y}{x^2+y^2},$$

the bulge reaches a maximum at $y = \pi/2$, so x should have a larger absolute value than y .

$$\frac{-\pi/2}{x^2+\pi^2/4} = \psi$$

$$-\pi/2 = (x^2 + \pi^2 / 4)\psi$$

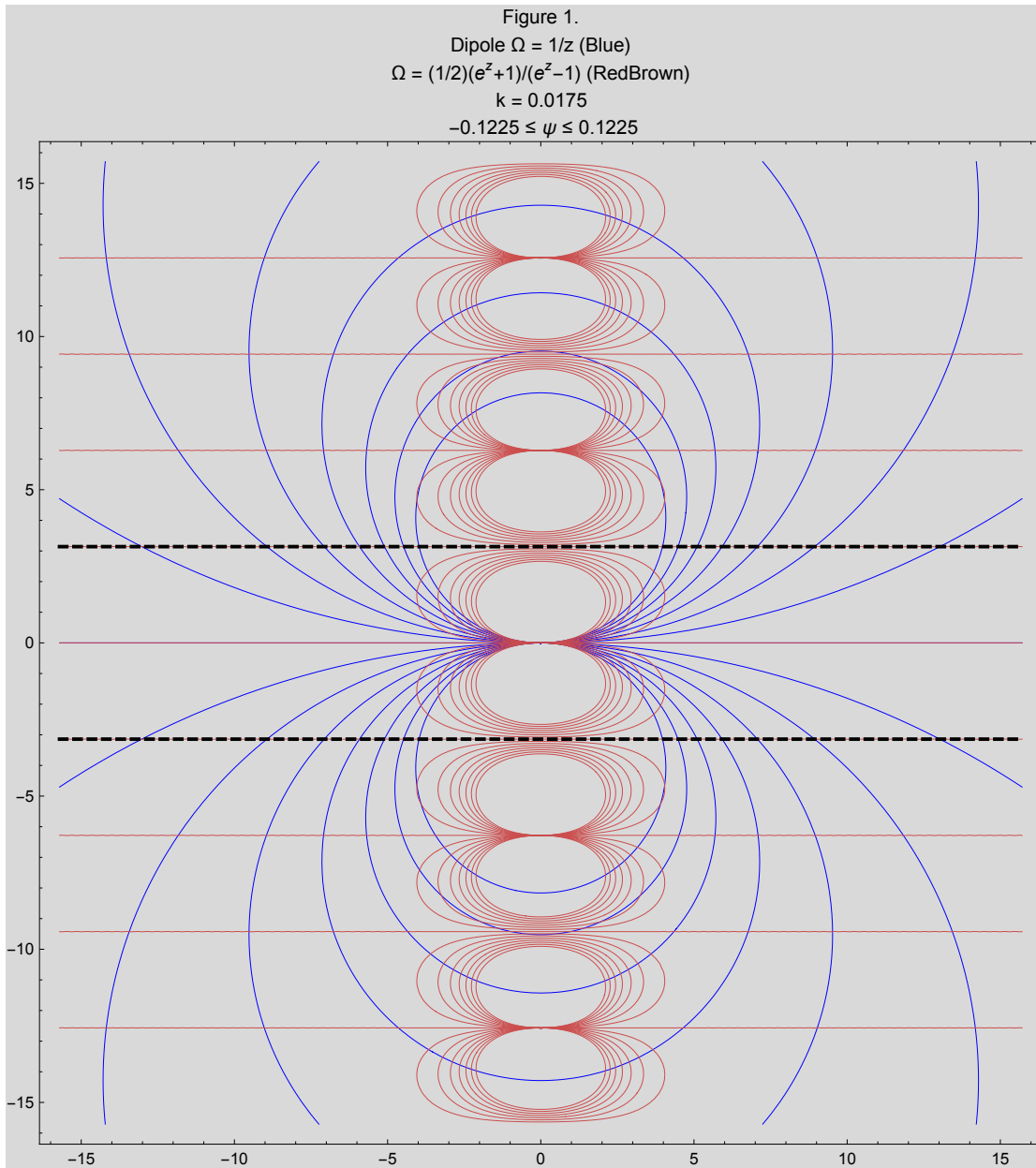
$$x^2\psi = -\pi/2 - (\pi^2 / 4)\psi$$

$$x^2 = -(\pi/(2\psi) + \pi^2 / 4)$$

$$x = \pm i \sqrt{\pi / (2\psi) + \pi^2 / 4}$$

I don't understand why we obtain a complex value for x , but $|x| > \pi/2$. If $\psi = \pm 0.1225$ (the range used in plotting Figure 1), $|x| = 3.910$, which is greater than $\pi/2 = 1.5708$. In other words, if we were to (somehow) confine the streamlines of Ω_A to the channel within the thick dotted lines, the value of $|x|$ at $\pm \pi/2$ would be about where it is for Ω_B , and within the channel of Figure 1, the blue plot would look very much like the red plot.

Figure 2, shows that at $y = \pi/2$ and $y = \pi$, velocity $|V|$, calculated from $V = -i \nabla \psi$ (p. 498), increases to a maximum at $x = 0$. This is probably true on every horizontal line. It is interesting that $|V|$ is higher on $y = \pi/2$ than on $y = \pi$, even though the crowded streamlines at $\pm\pi$ indicate higher flows.



$$\psi = -\frac{y}{x^2 + y^2}$$

$$\nabla\psi = \frac{2xy}{(x^2 + y^2)^2} + i\left(\frac{2y^2}{(x^2 + y^2)^2} - \frac{1}{x^2 + y^2}\right)$$

$$\bar{V} = -i\nabla\psi - i\left(\frac{2xy}{(x^2 + y^2)^2} + i\left(\frac{2y^2}{(x^2 + y^2)^2} - \frac{1}{x^2 + y^2}\right)\right)$$

$$V = i\left(\frac{2(xy)^*}{((x^*)^2 + (y^*)^2)^2} - i\left(\frac{2y^2}{(x^2 + y^2)^2} - \frac{1}{x^2 + y^2}\right)^*\right)$$

