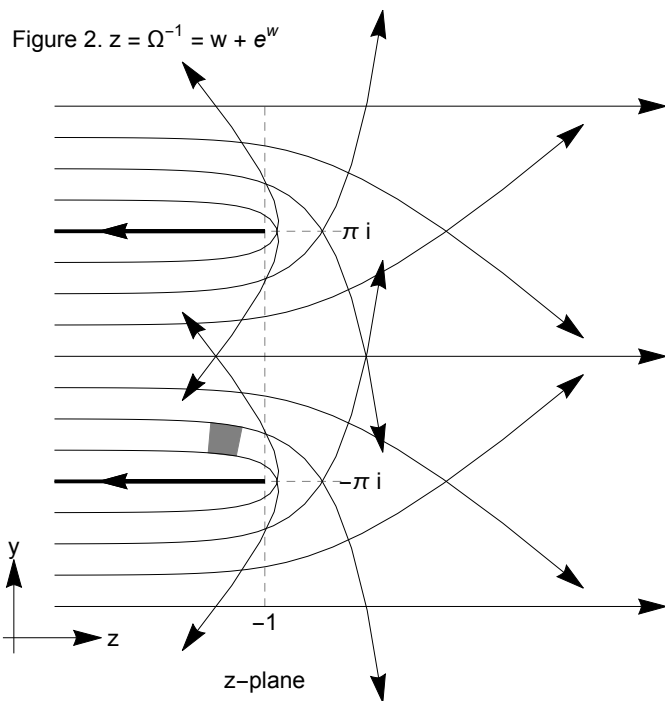
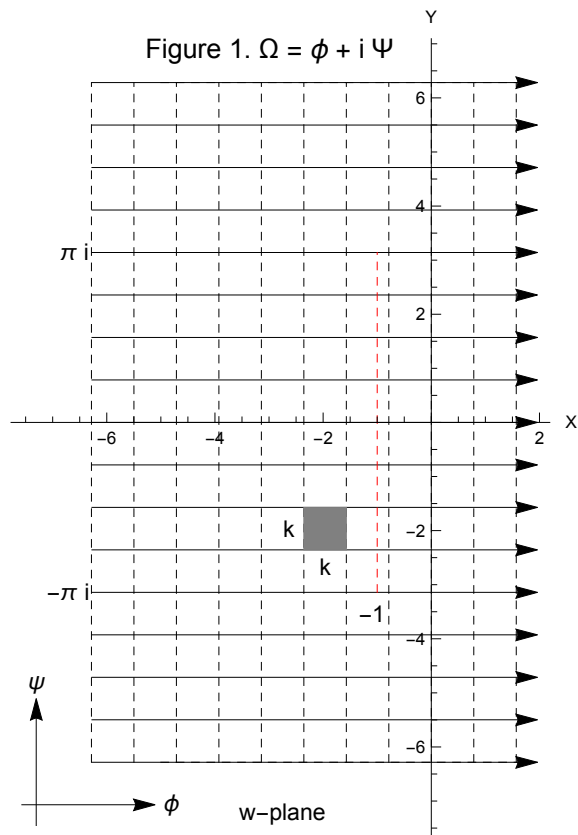


Needham, Chapter 11, Exercise 11, p. 507.
 G. Palmer, August 27, 2018, 2:20 pm PDT



To verify this mathematically means that “the picture”, which I take to be Figure 3 in the z plane, must depict flow for which the imaginary value of z stays within the imaginary constraints so long as the real value of z is ≤ -1 . When x exceeds -1 , z may wander from the channel established by the constraints.

Let $z = x + iy$.

$$\Omega^{-1} = w + e^w = z$$

$w = \phi(z) + i\psi(z)$, where ϕ and ψ are real analytic functions

$$\begin{aligned} z &= (\phi + i\psi) + e^{\phi+i\psi} \\ &= \phi + i\psi + e^{\phi}e^{i\psi} = \phi + i\psi + e^{\phi}(\cos\psi + i\sin\psi) \\ &= \phi + e^{\phi}\cos\psi + i(\psi + e^{\phi}\sin\psi) \end{aligned}$$

So

$$x = \phi + e^{\phi}\cos\psi \quad (1)$$

$$y = \psi + e^{\phi}\sin\psi \quad (2)$$

Ω^{-1} has a horizontal component of $\phi + e^{\phi}\cos\psi$ to which is added a vertical component $\psi + e^{\phi}\sin\psi$. We examine three situations: $\phi \leq -1$, $-1 < \phi \leq 0$, $\phi > 0$. Note that $\phi \leq -1$ is not the same as the constraint $\text{Re}(z) \leq -1$ on the channel, but beginning with the w -plane provides a way to examine the constraints on x and y in the z -plane.

$\phi < -1$

When $\phi < -1$, $e^{\phi} = 1 / e^{|\phi|}$, which implies from (1) that e^{ϕ} decreases exponentially with decreasing ϕ , so x will more closely approximate ϕ , and from (2), y will more closely approximate ψ . Hence, as Vasco noted in his answer, the streamline approaches its ψ value asymptotically with decreasing ϕ . As ϕ approaches -1 from below, e^{ϕ} goes to $1/e$, so $x = (-1 + (1/e)\cos\psi)$. At $\psi = 0$, $x = (-1 + 1/e) > -1$. When $\psi = \pm\pi$, $\phi - e^{\phi}$ increases with increasing ϕ up to -1 and decreases thereafter. Hence we see the reversal of the direction arrow on the thick lines $y = \pm\pi$.

$-1 < \phi \leq 0$

$e^{\phi} = 1 / e^{|\frac{1}{\delta}|}$, where $\phi = -1/\delta$ is the negative fractional exponent. Then, as ϕ approaches -1 from above, e^{ϕ} goes to $1/e$ and x goes to $(-1 + (1/e)\cos\psi)$, so x approaches its minimum value in this

range of ϕ at $(-1 - 1/e)$ when $\psi = \pm\pi$, and its maximum value at $(-1 + 1/e)$ when $\psi = 0$. Hence, the intersection in Figure 3 of the red line of potential with the horizontal lines $y = \pm\pi$ at $(-1 - 1/e)$ and the eastward bulge of the red line of potential near the origin up to $(-1 + 1/e)$. y goes to $\psi + (1/e) \sin \psi$. Sine has its maximum and minimum at $\pm\pi/2$. Since $\sin \psi = 0$ at $\psi = \pm\pi$, $y = \psi = \pm\pi$ at those values. These are the boundaries of the channel.

As ϕ approaches 0 from below, e^ϕ goes to 1 and x goes to $\cos \psi$, which has its maximum value 1 when $\psi = 0$ and its minimum value (-1) when $\psi = \pm\pi$. y approaches $(\psi + \sin \psi) = \pm\pi$ at $\psi = \pm\pi$. x approaches -1 at the boundaries.

$\phi > 0$

As ϕ increases indefinitely, e^ϕ goes to infinity. Both x and y can become very large. At large ϕ and x , y varies from 0 at $\pm\pi$ to $(\pm\pi/2 \pm e^\phi)$ within its principle branch where $-\pi \leq \psi \leq \pi$. At $\phi > 0$, x reverses direction.

In conclusion, when $\text{Re}(z) \leq -1$, the streamline approaches its ψ value asymptotically with decreasing ϕ within the boundaries $-\pi \leq \text{Im}(z) \leq \pi$. When $\text{Re}(z) > -1$, the diverging streamlines escape the boundaries $y = \pm\pi$. The divergence at $\text{Re}(z) = -1$ is gradual, except for the reversal of the boundary streamlines at $y = \pm\pi$ and $\text{Re}(z) > -1$, which happens when $\phi = 0$.

$$\begin{aligned} x &= \phi + e^\phi \cos \psi \\ -1 &= \phi + e^\phi (-1) \end{aligned}$$

Vasco pointed out that, at the horizontal boundaries, $x' = 0$ when $\phi = 0$, and x'' is negative, so x attains its maximum (-1) when $\phi = 0$.

When $\phi > 0$, both x and y can become very large, giving the appearance of a flow released from the boundaries of its channel. These considerations verify mathematically that the picture may be interpreted as the flow out of a channel $-\pi \leq \text{Im}(z) \leq \pi$, $\text{Re}(z) \leq -1$. It is problematic that the flow reverses at $x = -1$ on the horizontal boundaries. The fluid is theoretically incompressible and continuous in one direction (implied), so reversal of flow is a contradiction. The contradiction might be avoided by stipulating that $\phi \leq 0$.