

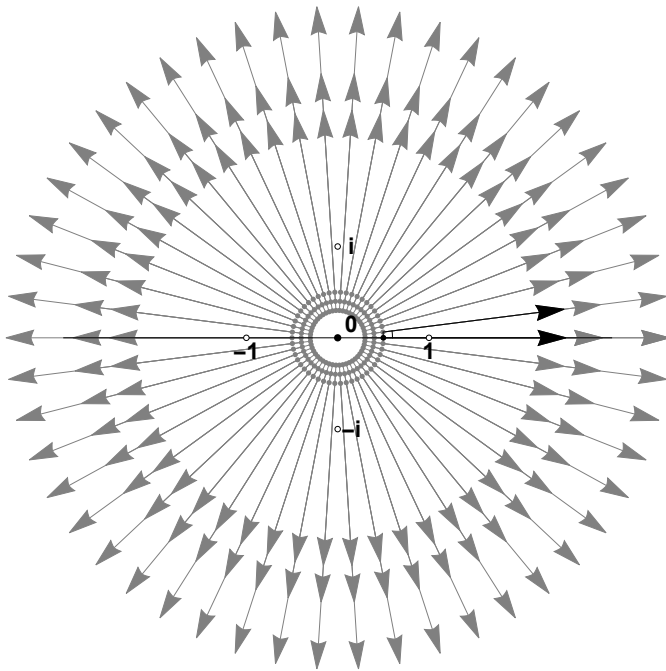


Figure 1.  $X(z) = 1/\bar{z}$

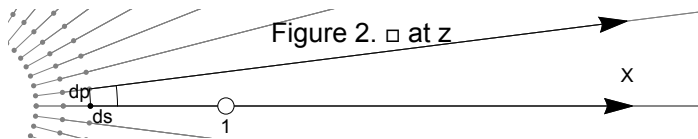


Figure 2.  $\square$  at  $z$

1. For each of the following vector fields  $X$  verify that the geometric formulae (5) and (6) yield the correct values for the divergence and for the curl.

(i)  $X = (1/\bar{z})$ .

### Divergence using (5)

This is a vector field apparently sourced at 0, but as  $z$  grows in absolute value, the flows into and out of  $R$  shrink. Plot a small quadrilateral  $R$  from a point  $z$  in the lower left corner, such as the point at the intersection of  $dp$  and  $ds$  in Figure 2. The line labeled  $ds$  is a segment of the ray, which is a streamline. The curve labeled “ $dp$ ” is a segment of a circle of radius  $r$  and curvature  $\kappa_P$ . This segment is orthogonal to  $ds$ . Then, by (5),

$$\nabla \cdot X = \partial_s \left| \frac{1}{\bar{z}} \right| + \kappa_P \left| \frac{1}{\bar{z}} \right|$$

$$\begin{aligned}
&= \partial_r \frac{1}{r} + \kappa_P \frac{1}{r} = -\frac{1}{r^2} + (\kappa_P)^2 \\
&= -(\kappa_P)^2 + (\kappa_P)^2 = 0
\end{aligned}$$

### Curl using (6)

Since the top and bottom of R are rays, which have no curvature,  $\kappa_S = 0$ . It also appears that  $\partial_p \left| \frac{1}{z} \right| = 0$ , because p depends on  $\theta$ , which is orthogonal to r, so the projection of X onto dp vanishes. By (6)

$$\begin{aligned}
\nabla \times \mathbf{X} &= -\partial_p \left| \frac{1}{z} \right| + \kappa_S \left| \frac{1}{z} \right| \\
&= -\partial_\theta \frac{1}{r} + 0 \cdot \frac{1}{r} \\
&= -0 + 0 = 0
\end{aligned}$$

Thus, it appears that both divergence and curl vanish on X when calculated from a geometric form.

**Verify (5) and (6) for  $\mathbf{X} = \frac{1}{z}$**  using (3) and (4)

Instead of plotting segments of a circle for dp and streamlines for ds, we plot a square centred at z, as in Figure [4], p. 477.

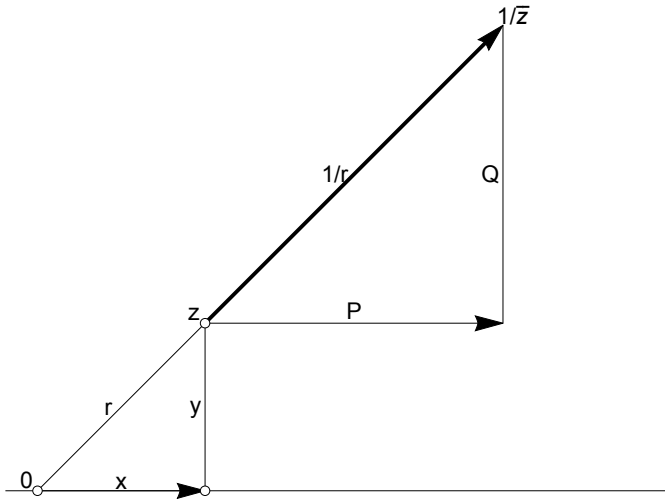


Figure 3. P from similar triangles.

**Verification of flux,  $\mathbf{X} = (1/z)$ .** P and Q are the horizontal and vertical projections of X at z. Given the notation introduced on p. 27, they are real numbers. In Figure 2, the gray dots at the tail of the

field vectors represent values of  $z$ . The gray arrows represent field vectors ( $1/\bar{z}$ ), which extend in the same direction as  $z$ . One can see that for these values of  $z$  inside the unit circle, the field vectors are longer than the  $z$  vectors. Let  $z = x + y i$ . Then let  $r = |z| = \sqrt{x^2 + y^2} < 1$ . In Figure 3, the triangles have three equal angles, so they are similar. We can write the ratios

$$\frac{x}{r} = \frac{P}{(1/r)} \quad \text{so} \quad P = \frac{x}{r^2} = \frac{x}{x^2 + y^2}$$

$Q = \frac{y}{x^2 + y^2}$  is found in the same way. These correspond to the values of  $P$  and  $Q$  in the analogous problem on p. 479, but leaving aside the coefficient  $S/2\pi$ . Vasco (personal communication) also has a nice method of finding  $P$  and  $Q$ :

$$\frac{1}{\bar{z}} = \frac{z}{z\bar{z}} = \frac{z}{|z|^2} = \frac{x+iy}{x^2+y^2},$$

from which one can read off the values of the projections in the  $x$  and  $y$  directions.

Then take the partials of  $P$  and  $Q$ .

$$\begin{aligned} \nabla \cdot \mathbf{X} &= \partial_x P + \partial_y Q \\ &= \partial_x P(z) + \partial_y Q(z) \\ &= \partial_x [x/(x^2 + y^2)] + \partial_y [y/(x^2 + y^2)] \\ &= \partial_x [x (x^2 + y^2)^{-1}] + \partial_y [y (x^2 + y^2)^{-1}] \quad \text{use Leibnitz} \\ &= [1 (x^2 + y^2)^{-1} + x (-(x^2 + y^2)^{-2} 2x)] + [1 (x^2 + y^2)^{-1} + y (-(x^2 + y^2)^{-2} 2y)] \\ &= \frac{1}{r^2} - 2x^2 \frac{1}{r^4} + \frac{1}{r^2} - 2y^2 \frac{1}{r^4} \quad \text{replace } (x^2 + y^2)^{1/2} \text{ with } r \\ &= \frac{2}{r^2} - 2r^2 \frac{1}{r^4} = 0 \end{aligned}$$

This verifies that the local flux vanishes on  $\mathbf{X} = 1/\bar{z}$ . Calculation on the web site <https://www.derivative-calculator.net/> gave

$$\frac{x^2 - y^2}{(x^2 + y^2)^2} + \frac{y^2 - x^2}{(x^2 + y^2)^2} = 0$$

which does verify the calculation from geometry.

**Verification of curl.**

$$\begin{aligned}
\nabla \times X &= \partial_x Q - \partial_y P \\
&= \partial_x (y (x^2 + y^2)^{-1}) - \partial_y (x (x^2 + y^2)^{-1}) \\
&= 0 - y (x^2 + y^2)^{-2} 2x - [0 - x (x^2 + y^2)^{-2} 2y] \\
&= -xy (x^2 + y^2)^{-2} + xy (x^2 + y^2)^{-2} = 0
\end{aligned}$$

This verifies that local curl vanishes on  $X = 1/\bar{z}$ .

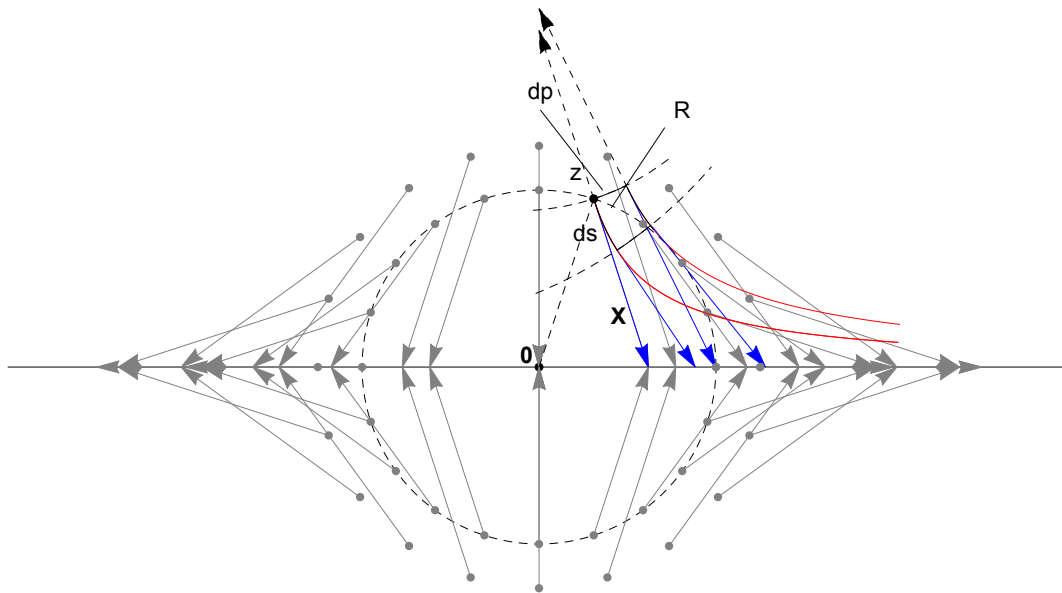


Figure 4.  $X(z) = \bar{z}$

● q

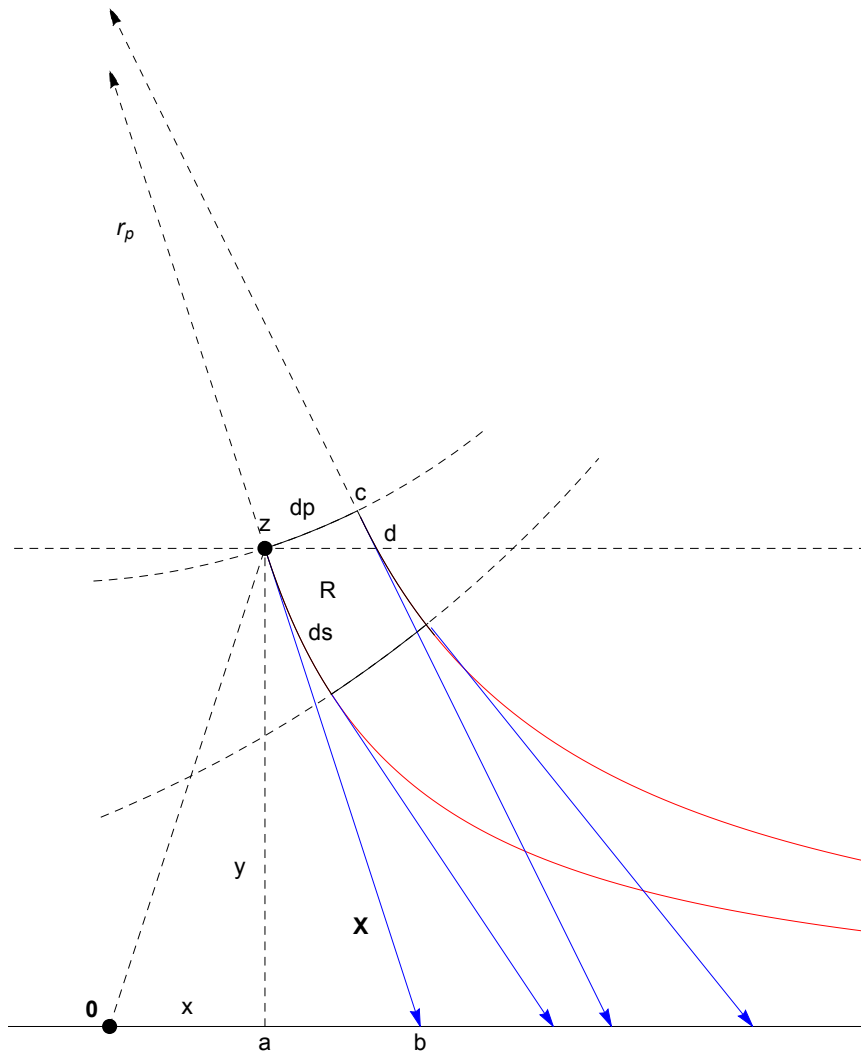
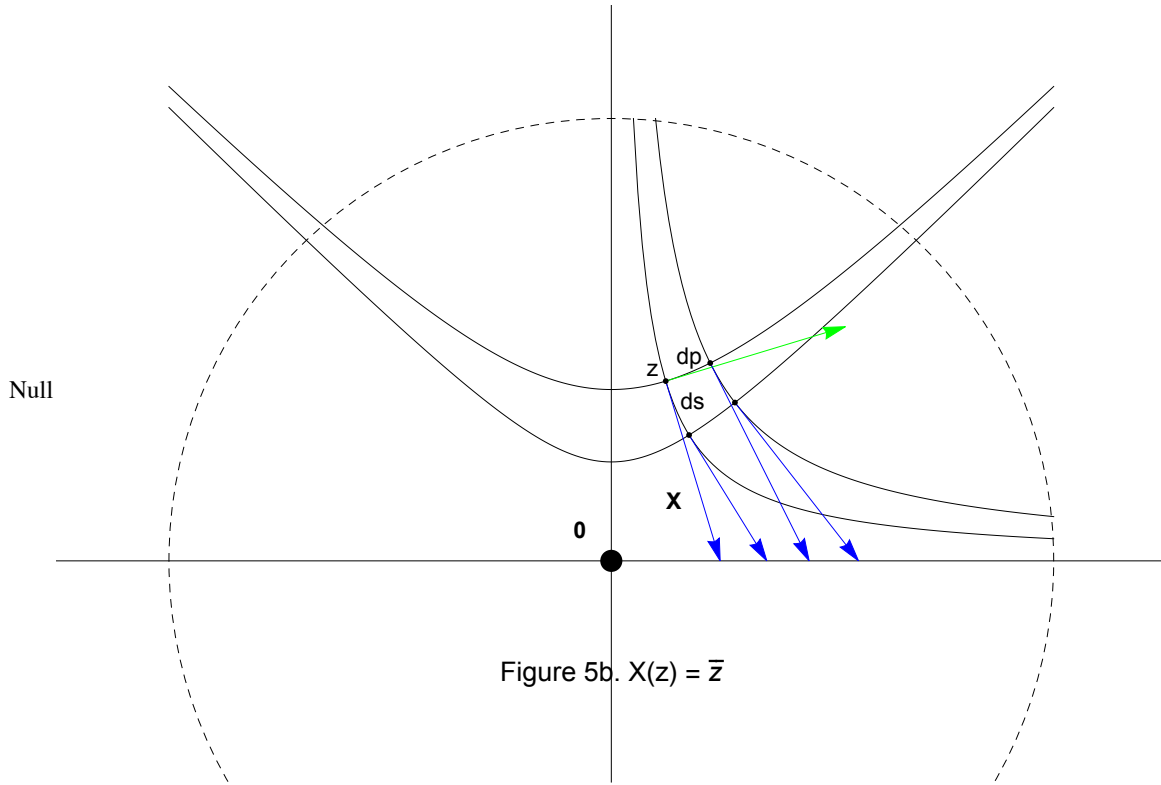


Figure 5. R at z



(ii)  $X = \bar{z}$ .

Figure 4 shows field vectors pointing towards and touching the real line. Compare [13], p. 490. The side labeled  $dp$  is plotted perpendicular to the flow vector  $X$ . It appears that the flow increases with  $|z|$ . Though it is not revealed in this diagram, it must be the case that the flow exiting  $R$  equals that entering, so the divergence should be 0. On p. 487, [11], the last term describes a vector field that is irrotational and sourceless, meaning it has neither divergence nor curl. That field is  $\bar{z}$ . Let  $r = \sqrt{x^2 + y^2} = |z| = |\bar{z}| = |X|$ . Then  $\kappa = 1/r$ .

### Divergence using (5).

Figure 5 reveals that with movement on  $ds$ ,  $|X|$  shrinks by nearly the same amount. As  $R$  shrinks,  $ds$  looks more like the chord on  $X$  cut out by the intersection with the opposite side to  $dp$ . The ratio goes to -1. Since  $\partial_s |X| = -1$ , it appears that in order to get the correct result of zero divergence, we must have that  $r_p = r$ . From Figure [8], p. 484, depicting the vector field  $|z|$  we can see that the vectors of length  $b$  emanate from the inner circle of radius  $b$ . The speed of the vectors entering area  $K$  equals  $r$  at the point of entry. From this we deduce that the blue  $X$  vectors in figures 4 and 5 emanate from points on  $dp$  where the radius of curvature  $r_p = r$ .

$$\nabla \cdot X = \partial_s |X| + \kappa_p |X|$$

$$= \partial_s r + \frac{1}{r_p} \mathbf{r}$$

$$= -1 + 1 = 0$$

We now try the more algebraic and precise approach used by Vasco. The results are plotted in Figure 5b. We will only look at the first quadrant. The tangents of a streamline run parallel to it. For  $X = \bar{z} = x - iy$ , the tangent is  $-y/x$ . Then the streamline has the equation  $dy/dx = -y/x \implies x dy + y dx = 0 \implies d(xy) = 0 \implies xy = c \implies y = c/x$ , which we use to plot two hyperbolas in the first quadrant. Then the orthogonal to the streamline has the negative inverse tangent  $dy/dx = x/y$ , from which we derive  $y dy - x dx = 0 \implies y^2 - x^2 = b$ , which we use to plot two hyperbolas symmetrical about the  $y$  axis. The two equations  $y = c/x$  and  $y^2 - x^2 = b$  can be solved to obtain the points of intersection (Figure 5b) from which the instances of  $X$  are plotted in blue.  $X(z)$  is plotted from the intersection point designated as  $z$ . A tangent to the orthogonal curve at  $z$  is plotted in green.

Vasco has

$$ds/dx = \{1 + (y')^2\}^{1/2} = \{1 + y^2/x^2\}^{1/2} = \pm(r/x)$$

and

$$ds/dy = \{1 + (1/y')^2\}^{1/2} = \{1 + x^2/y^2\}^{1/2} = \pm(r/y).$$

He uses the signed curvature  $\kappa = y''/\{1 + (y')^2\}^{3/2}$  to calculate  $\kappa_P$ . The slope of the orthogonal trajectory is  $x/y$ .

$$y' = x/y$$

$$y'' = (1 \cdot y - x y') / y^2 \quad (\text{implicit derivative and quotient rule})$$

$$= (y - x(x/y)) / y^2 \quad (\text{substitute for } y')$$

$$= (y^2 - x^2) / y^3 \quad (\text{multiply top and bottom by } y)$$

$$\kappa_P = \{(y^2 - x^2) / y^3\} / \{1 + x^2 / y^2\}^{3/2}$$

$$= \{(y^2 - x^2) / y^3\} / \{(x^2 + y^2) / y^2\}^{3/2}$$

$$= \{(y^2 - x^2) / y^3\} (r^3 / y^3)$$

$$= (y^2 - x^2) / r^3$$

Then

$$\begin{aligned}
 \nabla \cdot \mathbf{X} &= \partial_s |X| + \kappa_P |X| \\
 &= \partial_s r + \kappa_P r \\
 &= [\partial_x r \cdot (dx/ds) + \partial_y r \cdot (dy/ds)] + [(y^2 - x^2)/r^3] r \\
 &= [\partial_x r \cdot (x/r) + \partial_y r \cdot (-y/r)] + (y^2 - x^2)/r^2 \\
 &= [(x/r) \cdot (x/r) + (y/r) \cdot (-y/r)] + (y^2 - x^2)/r^2 \\
 &= (x^2 - y^2)/r^2 + (y^2 - x^2)/r^2 \\
 &= 0
 \end{aligned}$$

**Verify using (3) with a square box centred at z.** Then at z, where  $\bar{z} = x - yi$ ,  $P = x$  and  $Q = -y$ .

$$\begin{aligned}
 \nabla \cdot \mathbf{X} &= \partial_x P + \partial_y Q & [Q = \text{Im}(-yi)] \\
 &= \partial_x x + \partial_y (-y) \\
 &= 1 - 1 = 0
 \end{aligned}$$

This verifies (5) using (3).

**Curl using (6).** The change in magnitude of  $|\bar{z}|$  increases with p, as shown by the increasing length of the blue arrows in Figures 4 and 5, but the difference between the flow into and out of R remains 0 on p.

$$\begin{aligned}
 \kappa_S &= y''/\{1 + (y')^2\}^{3/2} \\
 y' &= -y/x \\
 y'' &= (-y' x - (-y) \cdot 1) / x^2 \\
 &= ((y/x)x + y) / x^2
 \end{aligned}$$

$$= 2y/x^2$$

$$\kappa_S = (2y/x^2)/\{1 + (-y/x)^2\}^{3/2} \quad (\text{by substitution for } y'' \text{ and } y')$$

$$dp/dx = \{1 + (y')^2\}^{1/2} = \{1 + x^2/y^2\}^{1/2} = \pm(r/y)$$

$$dp/dy = \{1 + (1/y')^2\}^{1/2} = \{1 + y^2/x^2\}^{1/2} = \pm(r/x)$$

$$\nabla \times \mathbf{X} = -\partial_p |\bar{z}| + \kappa_S |\bar{z}|$$

$$= -\partial_p r + \kappa_S r$$

$$= -[\partial_x r \cdot (y/r) + \partial_y r \cdot (x/r)] + [(2y/x^2)/\{1 + (-y/x)^2\}^{3/2}] r$$

$$= -[(x/r)(y/r) + (y/r)(x/r)] + (2y/x^2) / \left[ \{(x^2 + y^2)/x^2\}^{3/2} \right] r$$

$$= -2xy/r^2 + [(2y/x^2)(r^2/x^2)^{3/2}] r$$

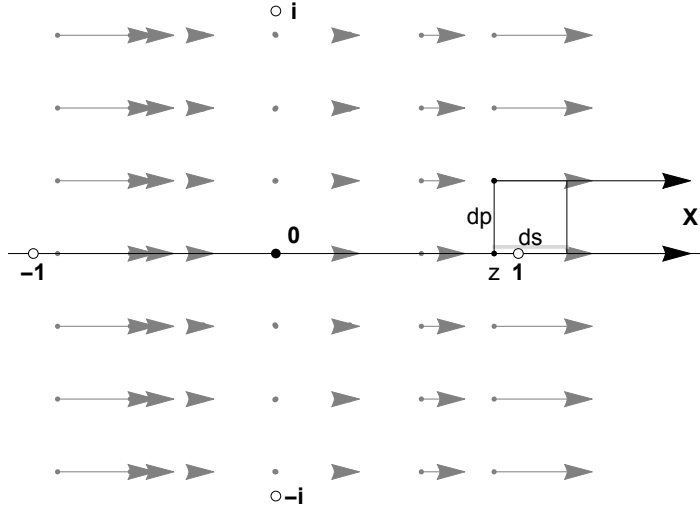
$$= -2xy/r^2 + [(2y/x^2)(r^3/x^3)] r$$

$$= -2xy/r^2 + 2xy/r^2 = 0$$

**Verify curl from (6) using (3).**

$$\nabla \times \mathbf{X} = \partial_x y - \partial_y x$$

$$= 0 - 0 = 0$$

Figure 6.  $X = x^2$ 

(iii)  $X = x^2$ , where  $z = x + iy$

The vector field  $x^2$  consists of vectors parallel to the positive x axis and pointing East (Figure 6). Note that in Figure 6 the gray field vectors are plotted at half their actual lengths. Thus, the small box with the two field vectors of actual length emanating from the east side does not illustrate the correct divergence. The vectors of the vector field are all parallel to the real line, so the orthogonal trajectories are vertical lines. Then, since all the P lines are straight and vertical, and all the X vectors are straight and horizontal,  $\kappa_P = 0$  and  $\kappa_S = 0$ ,  $\partial_s = \partial_x$ ,  $\partial_p = \partial_y$ .

**Divergence using (5):**

$$\nabla \cdot X = \partial_x |x^2| + \kappa_P |x^2| = \partial_x x^2 + 0 \cdot x^2 = 2x$$

**Verifying divergence using (3):**

The projection P of  $x^2$  onto a horizontal line through z equals  $x^2$ . There is no projection of  $x^2$  onto a vertical line through z. Therefore,  $Q = 0$ . In other words  $X = x^2 = x^2 + 0i = P + Q \equiv (P \ Q)^t$ .

$$\nabla \cdot X = \partial_x P + \partial_y Q = \partial_x x^2 + \partial_y 0 = 2x$$

This verifies the calculation using (5).

**Curl using 4.**

$$\nabla \times X = -\partial_p |x^2| + \kappa_S |x^2| = -\partial_y x^2 + 0 \cdot x^2 = 0 + 0 = 0$$

**Validation of curl obtained with (6) using (4).**

$$\nabla \times X = \partial_x Q - \partial_y P = \partial_x 0 - \partial_y x^2 = 0 - 0 = 0$$

This verifies the calculation with (6).

(iv)  $X = y^2$ , where  $z = x + iy$ .

$X$  is real, so the vectors are all horizontal lines as in (iii), so the orthogonal lines are vertical. Then  $\kappa_P = 0$  and  $\kappa_S = 0$  as in (iii).

**Divergence applying (5).**  $\partial_s$  should be on the streamlines. Then the curvature, if any, on  $P$  is multiplied by  $|X|$ .

$$\nabla \cdot X = \partial_s y^2 + \kappa_P y^2$$

$$\partial_x y^2 + 0 y^2$$

$$= 0 + 0 = 0$$

**Validation with (3).**  $P = y^2$ ;  $Q = 0$ .

$$\nabla \cdot X = \partial_x y^2 + \partial_y 0 = 0 + 0 = 0$$

This verifies the calculation from (5).

**Curl applying (6).**  $\kappa_S = 0$ .

$$\nabla \cdot X = -\partial_p y^2 + \kappa_S y^2$$

$$\approx -\partial_y y^2 + 0 y^2$$

$$= -2y$$

**Validation of curl obtained with (6) using (4).**

$$\nabla \times X = \partial_x 0 - \partial_y y^2$$

$$= 0 - 2y = -2y$$

This verifies the calculation with (6).

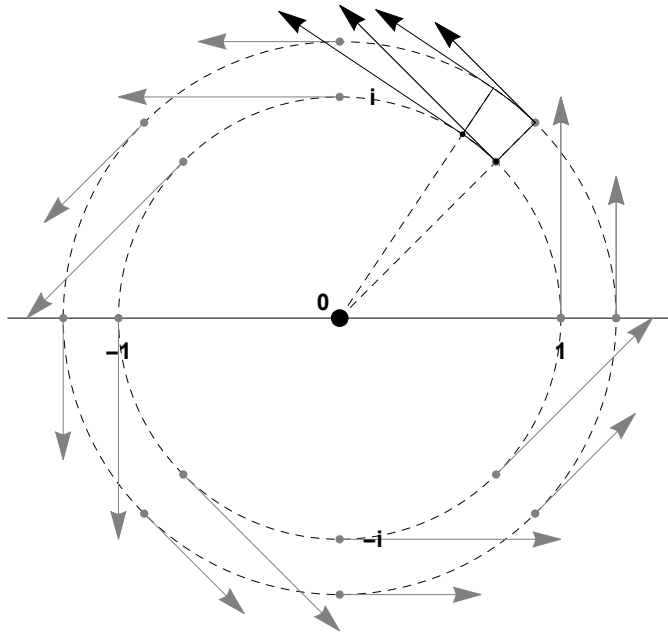


Figure 7. Field  $X(z) = i(1/r^2)e^{i\theta}$

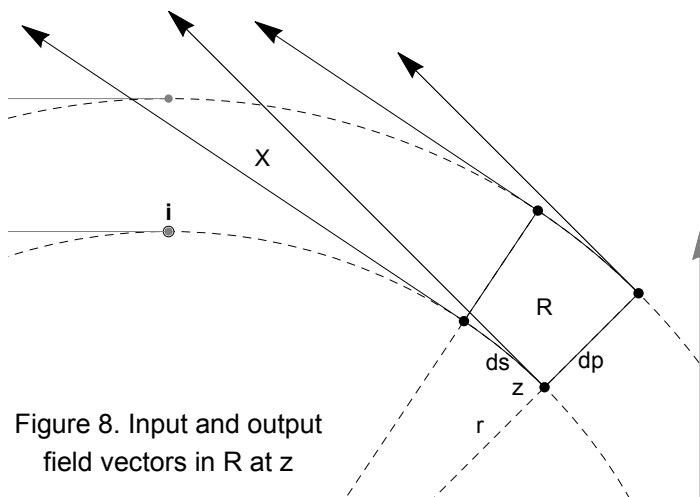


Figure 8. Input and output field vectors in  $R$  at  $z$

(v)  $X = i \left( 1 / r^2 \right) e^{i\theta}$ , where  $z = r e^{i\theta}$ .

This field can be rewritten  $\frac{iz}{r^3}$ . It is sourceless (i.e. vanishing flux with all simple loops in  $R$ , p. 473, S. -1) with constant curl much like that of  $\frac{iz}{2}$  in Figure [11]. The streamlines are circles, as shown by

the dashed lines in Figures 7 and 8.

**Divergence applying (5).**  $\kappa_S = (1/r)$ .  $\kappa_P = 0$ , because  $dp$  is a straight line orthogonal to the circular streamlines.  $ds$  is used to label a segment of the circumference spanned by some  $d\theta$ .

$$\partial_s \left| i \left( 1 / r^2 \right) e^{i\theta} \right| = \partial_s \left( 1 / r^2 \right) = 0, \text{ because } r \text{ is constant on } ds.$$

$$\begin{aligned} \nabla \cdot X &= \partial_s \left| i \left( 1 / r^2 \right) e^{i\theta} \right| + \kappa_P \left| i \left( 1 / r^2 \right) e^{i\theta} \right| \\ &= \partial_s \left( 1 / r^2 \right) + 0 \cdot \left( 1 / r^2 \right) \\ &= 0 + 0 \end{aligned}$$

**Validation of divergence obtained with (5) using (3).**

$$X = i \left( 1 / r^2 \right) (\cos \theta + i \sin \theta)$$

$$= \frac{i \cos \theta}{r^2} - \frac{\sin \theta}{r^2} = \frac{i x}{r^3} - \frac{y}{r^3}$$

$$P = i \left( 1 / r^2 \right) i \sin(\theta) = - \left( 1 / r^2 \right) \sin(\theta)$$

$$Q = \text{Im}[i \left( 1 / r^2 \right) \cos(\theta)] = \left( 1 / r^2 \right) \cos(\theta)$$

$$\nabla \cdot X = \partial_x P + \partial_y Q$$

$$\begin{aligned} &= \partial_x \left( - \left( 1 / r^2 \right) \sin(\theta) \right) + \partial_y \left( \left( 1 / r^2 \right) \cos(\theta) \right) \\ &= -\partial_x \left( \left( 1 / r^3 \right) y \right) + \partial_y \left( \left( 1 / r^3 \right) x \right) \\ &= -\partial_x \left( y \left( x^2 + y^2 \right)^{-3/2} \right) + \partial_y \left( x \left( x^2 + y^2 \right)^{-3/2} \right) \\ &= y \left( 3/2 \right) \left( x^2 + y^2 \right)^{-5/2} (2x) + x \left( -3/2 \right) \left( x^2 + y^2 \right)^{-5/2} (2y) \\ &= 3xy \left( x^2 + y^2 \right)^{-5/2} - 3xy \left( x^2 + y^2 \right)^{-5/2} = 0 \end{aligned}$$

This verifies the calculation using (5).

**Curl using (6).**

If this is correct, it was achieved by changing the model. Needham has

$$\nabla \times \mathbf{X} = -\partial_p |\mathbf{X}| + \kappa_S |\mathbf{X}|.$$

I could only validate (6) with (4) by writing

$$\nabla \times \mathbf{X} = \partial_p |\mathbf{X}| + \kappa_S |\mathbf{X}|.$$

Figure 8 makes it appear that the vectors on the inner circle  $S$  are longer than those on the outer circle  $S'$ . If one begins at  $z$  and traverses  $dp'$  (opposite side to  $dp$ ),  $ds'$  (the  $ds$  segment of the outer circle,  $S'$ ),  $dp$ , and  $ds$  in a counter clockwise direction, then the difference between the two vectors on  $ds$  is larger than that of the two vectors on  $ds'$  because the vectors are longer, but the vectors point opposite to the direction of traversal, so the curl is negative. This is a different situation from that illustrated in [5], where the curl on  $ds$  and  $ds'$  had the same orientation. Therefore, I write the  $\partial_p |\mathbf{X}|$  as a positive value. In his answer, Vasco calculated the direction of  $dp$  as the direction of  $i\mathbf{X}$ , so that the orthogonal vectors point towards the center. Then  $-\partial_p = -\partial_{-r}$ , which would change the sign to positive, as I have done here.

$$\begin{aligned} \nabla \times \mathbf{X} &= \partial_p \left| i \left( 1 / r^2 \right) e^{i\theta} \right| + \kappa_S \left| i \left( 1 / r^2 \right) e^{i\theta} \right| \\ &\approx \partial_r \left( 1 / r^2 \right) + (1 / r) \left( 1 / r^2 \right) \\ &= \partial_r \left( r^{-2} \right) + 1 / r^3 \\ &= -2 / r^3 + 1 / r^3 \\ &= -1 / r^3 \end{aligned}$$

**Validation of curl obtained with (6) using (4).**

$$\mathbf{X} = i \left( 1 / r^2 \right) (\cos \theta + i \sin \theta)$$

$$= \frac{i \cos \theta}{r^2} - \frac{\sin \theta}{r^2} = \frac{ix}{r^3} - \frac{y}{r^3}$$

$$\mathbf{P} = - \left( y / r^3 \right)$$

$$\mathbf{Q} = \left( x / r^3 \right)$$

$$\nabla \times \mathbf{X} = \partial_x \mathbf{Q} - \partial_y \mathbf{P}$$

$$\begin{aligned}
&= \partial_x \left[ \left( x / r^3 \right) \right] - \partial_y \left[ - \left( y / r^3 \right) \right] \\
&= \partial_x \left[ x \left( x^2 + y^2 \right)^{-3/2} \right] + \partial_y \left[ y \left( x^2 + y^2 \right)^{-3/2} \right] \\
&= 1 \left( x^2 + y^2 \right)^{-3/2} + x \left( -3/2 \right) \left( x^2 + y^2 \right)^{-5/2} (2x) + \left( 1 \left( x^2 + y^2 \right)^{-3/2} + y \left( -3/2 \right) \right. \\
&\quad \left. \left( x^2 + y^2 \right)^{-5/2} (2y) \right) \\
&= 2 \left( x^2 + y^2 \right)^{-3/2} - 3 x^2 \left( x^2 + y^2 \right)^{-5/2} - 3 y^2 \left( x^2 + y^2 \right)^{-5/2} \\
&= 2 \left( x^2 + y^2 \right)^{-3/2} - 3 \left( x^2 + y^2 \right) \left( x^2 + y^2 \right)^{-5/2} \\
&= 2 \left( x^2 + y^2 \right)^{-3/2} - 3 \left( x^2 + y^2 \right)^{-3/2} \\
&= - \left( x^2 + y^2 \right)^{-3/2} \\
&= \frac{-1}{\left( x^2 + y^2 \right)^{3/2}} = - \frac{1}{r^3}
\end{aligned}$$

This validates the result obtained with (6).