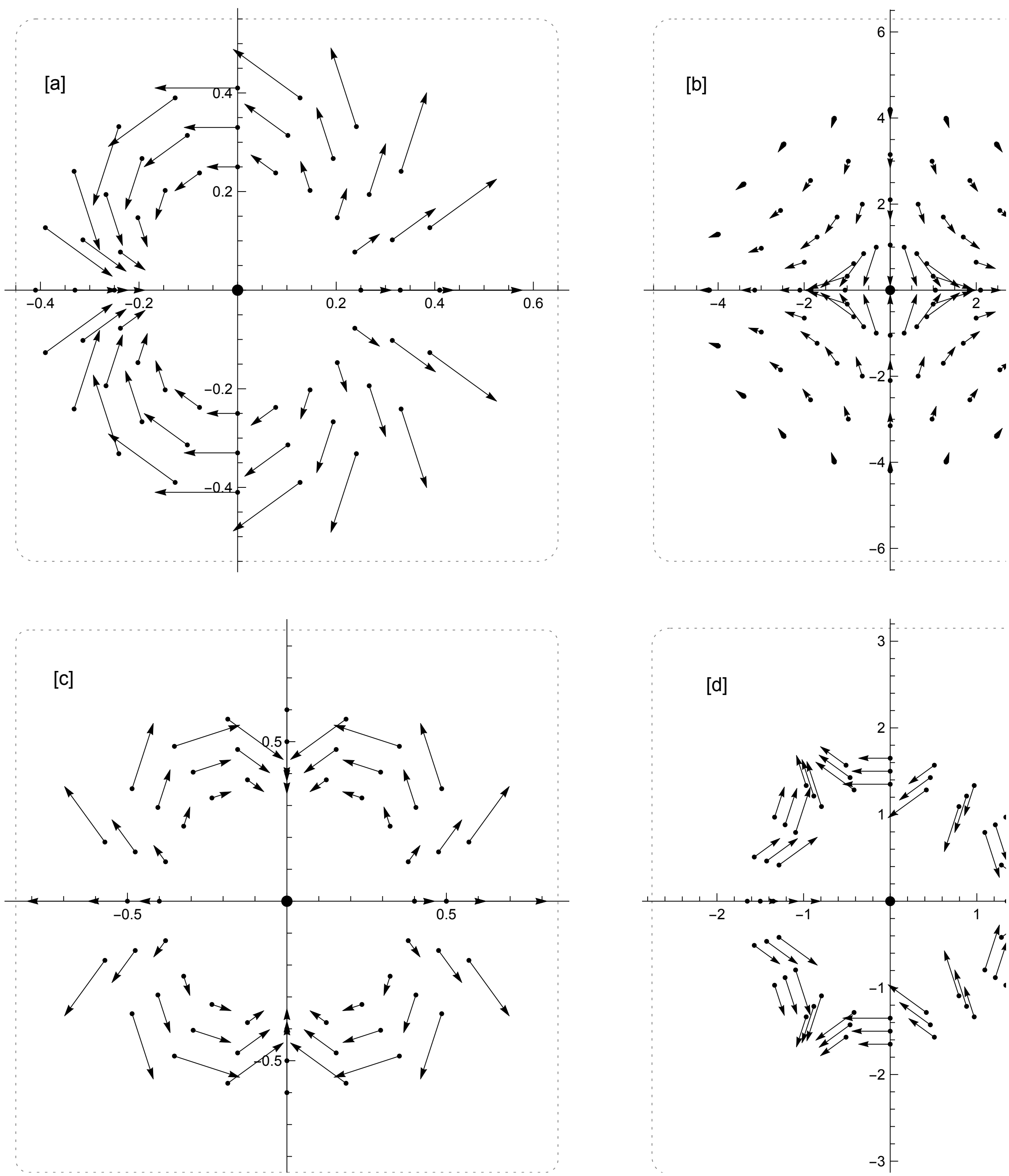
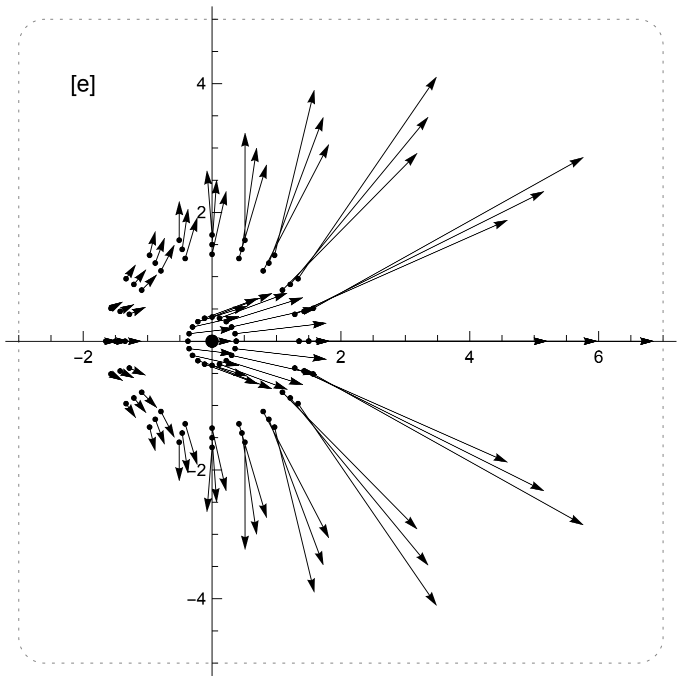
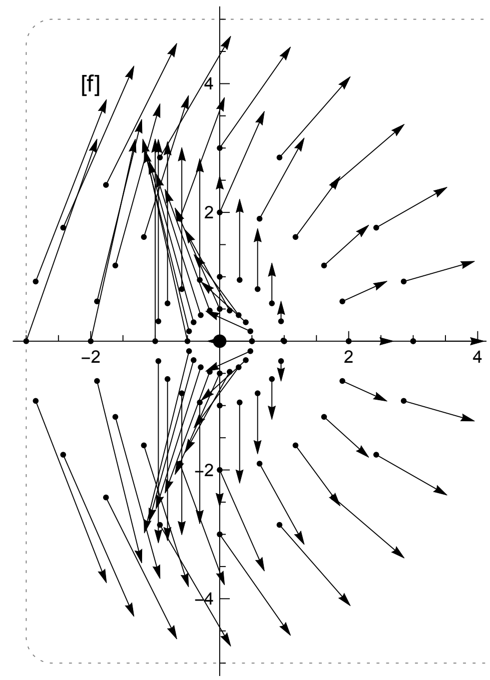
Notes on Needham, Chapter 10.
G. Palmer, July 20, 2018, 10 am PT

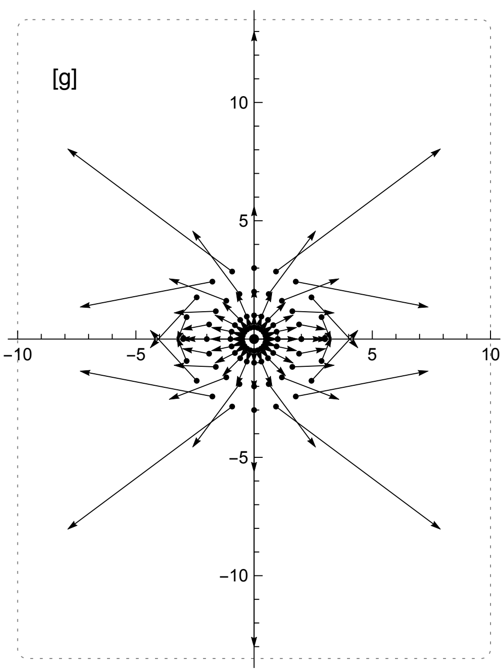
P. 450 *Try doing a vector sketch of some other powers, then compare them with accurate ones done by your computer. Also use the computer to examine the vector fields of e^z, log z, and sin z.*

Vector fields
[a] z^2, [b] $1/z$, [c] z^3, [d] $1/z^2$, [e] e^z, [f] $\log[z]$, [g] $\sin(z)$









Flow, force, irrotational, source and loop

It has been a mystery to me how one knows that a flow or force has or does not have a source. The information appears to be provided on these pages.

P. 453. The flow of water is described as incompressible and irrotational.

P. 454. Indicates that a vector field can be thought of as the velocity of flow and streamlines as

paths. Force fields are distinct from flows, but they are represented in the same way, so force is like velocity. The streamlines of force fields are the lines of force. The streamlines may be rays. Flux is flow (or force?) across a curve and orthogonal to the curve.
P. 455 The idea of flux across a loop appears to introduce the idea of a source.

$$S = 2\pi\, r\, |V|, \qquad V = \frac{S}{2\pi}(1/\bar{z})$$

P. 473-474 (Chap. 11). A flow is sourceless in a region if all simple loops in the region have vanishing flux. This must mean that the loops do not enclose a singular point.

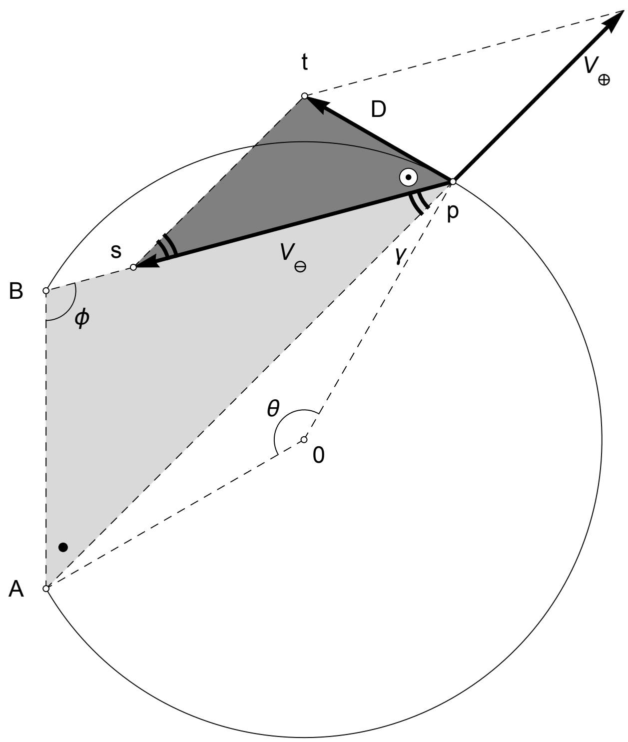


Figure 1. Facsimile of [4b], p. 455

P. 456. *We will now show geometrically that as claimed in [3], the net force at p is tangent to the circle through A, p, and B. Consider [4b]. It is easy to see [exercise] that D will be tangent to the circle if and only if the angles marked •* and ⊙ are equal, so this is what we must demonstrate.

We have plotted Figure 1 using the equations for $V_\oplus$ and $V_\ominus$ provided on p. 455, letting S = 1. It is necessary to use the equations in order to obtain the correct lengths of $V_\oplus$ and $V_\ominus$, without which the net force (D = $V_\oplus + V_\ominus$) will not be tangent ot the circle at p.

To see Needham’s thinking, we should show that

$$\frac{\text{ts}}{\text{ps}} = \frac{|V_\oplus|}{|V_\ominus|} = \frac{\text{Bp}}{\text{Ap}}$$

This establishes that the two shaded triangles are similar. Then • = ⊙. Then we show that • = ⊙ implies that D is tangent to the circle.

Since ts is parallel to $V_\oplus$ by the equal angles at tsp and spA, we can write

$$\frac{\text{ts}}{\text{ps}} = \frac{|V_\oplus|}{|V_\ominus|}$$

But

$$V_\oplus = \frac{S}{2\pi}\left(\frac{1}{\overline{z}-\overline{A}}\right)$$

$$V_\ominus = \frac{S}{2\pi}\left(\frac{1}{\overline{z}-\overline{B}}\right)$$

implies that

$$\frac{V_\oplus}{V_\ominus} = \frac{\overline{z}-\overline{B}}{\overline{z}-\overline{A}} = \frac{\text{Bp}}{\text{Ap}}$$

so we have established that

$$\frac{\text{ts}}{\text{ps}} = \frac{|V_\oplus|}{|V_\ominus|} = \frac{\text{Bp}}{\text{Ap}}$$

which means that two sides of the triangles are proportional and the angles between the two sides of each are equal. The shaded triangles are therefore similar, which establishes that • = ⊙ and we have technically completed the exercise.

Now we want to show that D must be tangent to the circle. If D is tangent, then ⊙ + // + $\gamma = \pi/2$. By an apparently well known relation, if two triangles share a chord, one triangle having a vertex at the circle origin (A0p) and the other having a vertex on the circle (ABp) (Vasco, Forum), we can write:

$$\theta = 2\pi - 2\phi$$

Then we can write, using // for the double arc angle symbol,

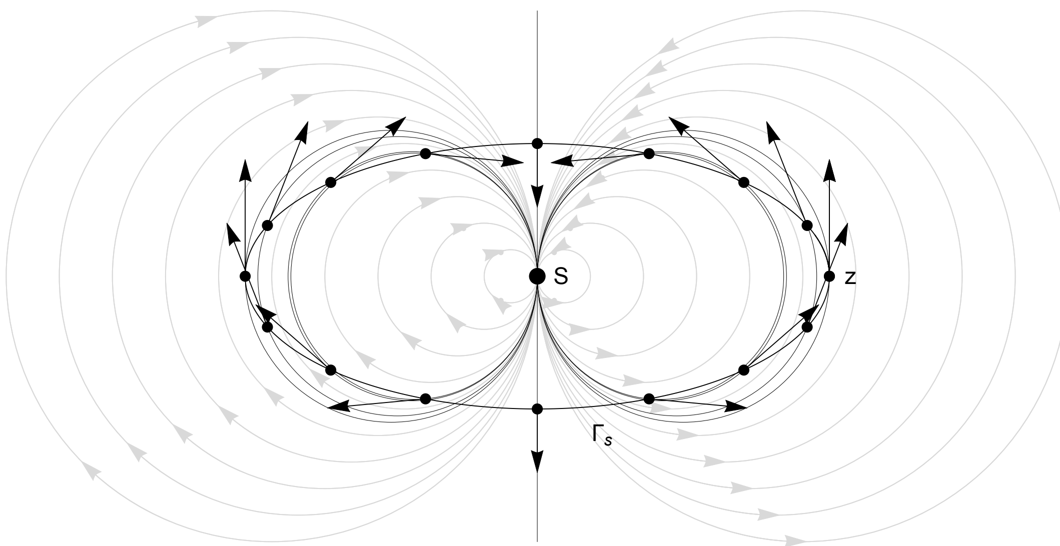
• = π - (ϕ + //)
$\gamma = (\pi-\theta)/2$
⊙ + // + $\gamma = \pi/2$
  = $\pi/2$ - // - γ

$= \pi/2$ - // - $(\pi-\theta)/2$
= - // + $\theta/2$ = -// + $(2\pi-2\phi)/2$
= -// + π - ϕ = •

We have shown that • = ⊙ if D is tangent to the circle. Since we know already that • = ⊙ by the similarity of the shaded triangles, then D must be tangent. So D is tangent if and only if the angles marked • and ⊙ are equal.

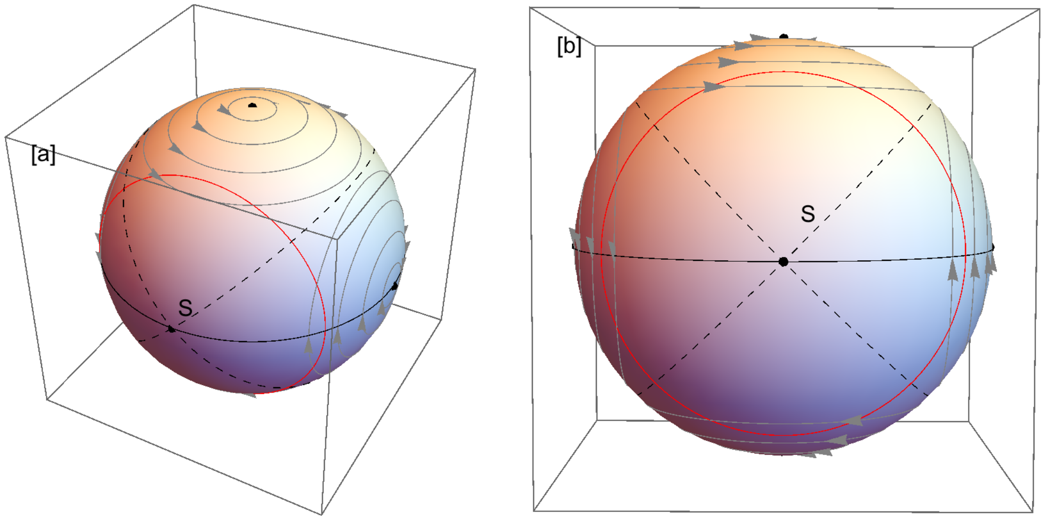
P. 457 [5] Dipole

Vectors on an ellipse have been added to the dipole.



The figure represents a vector field induced by the dipole shown in [5], p. 457 acting on an elliptical curve Γ_S. As z traverses Γ_S in a counter-clockwise direction, the vector arrow rotates continously in the counter-clockwise direction about an imagined zero point at the vector base, resulting in index $\mathcal{I}$[S] = +2. The four thin black circles in each dipole and the central vertical line are streamlines intersected by z at 16 arbitrary points on Γ_S. Two of these circles are very close together between directed circles 4 and 5 on both sides. The outer two lie between directed circles 5 and 6. Note that the vertical line can also be thought of as a circle.

Test of Poincaré-Hopf Theorem, p. 462



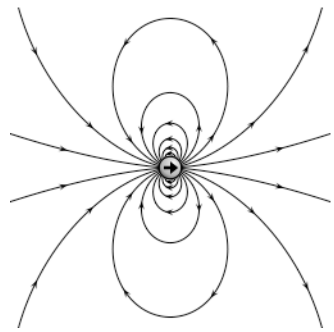
In the two views [a] and [b] of the sphere above, groups of circles are placed on the sphere representing four vortices (right, top, left, bottom) each of which is analogous to that in the second plot, top row, of [5], p. 457. Each of the vortices on the sphere has a singular point at the center. Viewing from the outside of the sphere, let z traverse counterclockwise on any one of the streamlines in a single vortex to produce a vector field with index $\mathcal{I}[S_i] = +1$, no matter the direction of the streamlines. The indices of the four vortices sum to +4. This appears to contradict the Poincaré-Hopf theorem, which states that the sum of the indices of any vector field on the surface of a topological sphere is 2. But in fact, we must also consider the indices of the remaining two singular points. On p. 463, Needham wrote "sum the indices of the singular points."

Also visible is a simple crosspoint S where two great circles (dashed lines) outside the vortices intersect to form a saddle on the sphere analogous to the first plot in [5], p. 457. The pattern is symmetrical front to back. That is, the view on the back side of the sphere is identical to that on the front, so there are two crosspoints, i.e. two more singular points. Let z traverse the red circle in the counterclockwise direction. Draw vectors tangent to the gray directed streamlines of the vortices at the 8 points where they are intersected by Γ_s. As z enters a new vortex, the rotation of each vector with respect to the preceding vector is always negative. When we take into account the fact that the gray circles are just samples from an infinite number of directed streamlines, and the fact that the circular streamlines can be distorted (p. 464, last ¶) to look more like the streamlines in the simple crosspoint in [5], we conclude that a complete traversal of z produces a complete vector rotation of -1 just as the simple crosspoint does in [5]. Since there are two saddle points, the sum of indices for them is -2. Now all the indices can be summed as on p. 463, top.

$$\mathcal{I}(\text{vortices}) + \mathcal{I}(\text{saddles}) = +4 - 2 = 2$$

This agrees with the Poincaré-Hopf Theorem.

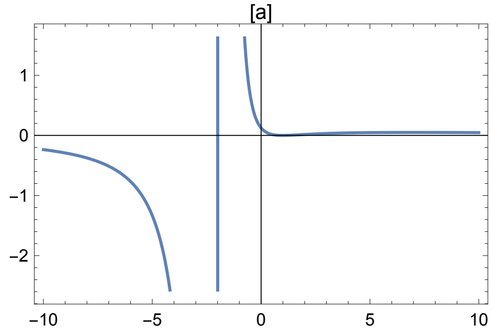
Needham says "Intuitively, we may picture such a vector field as the velocity of a fluid that is flowing over S" (p. 463). There seem to be no constraints on the types of flow that might be induced. In some cases, such as [11a], the vortices could be imagined as created by a drenched sphere spinning on one or more axes. In the present example, one can think of four paint mixers of the spinning paddle type applied to create the four vortices. What would create the dipole in [11b]? It appears to be a "point dipole" in which the distance between singular points approaches 0. Wikipedia defines a point electric dipole as follows: "A point (electric) dipole is the limit obtained by letting the separation tend to 0 while keeping the dipole moment fixed." An electron has a magnetic dipole, so it might serve as a good example of a point dipole.



https://en.wikipedia.org/wiki/Dipole
https://upload.wikimedia.org/wikipedia/commons/thumb/8/81/VFPt_dipole_point.svg/250px-VFPt_dipole_point.svg.png

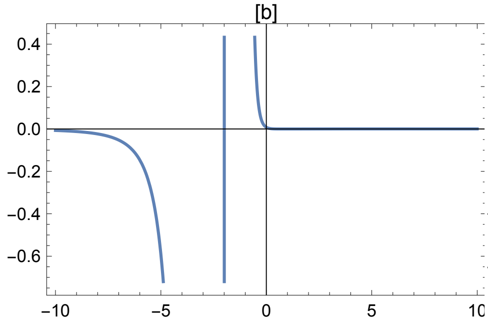
P. 458, bottom, graphs of two functions

[a]



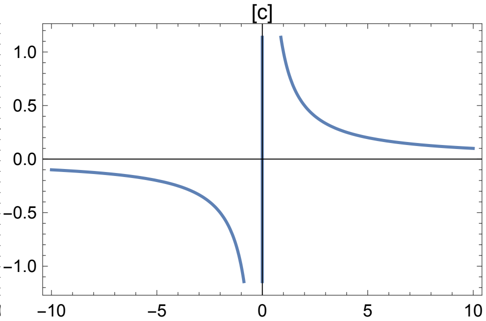
$(x-1)^2/(x+2)^3$

[b]

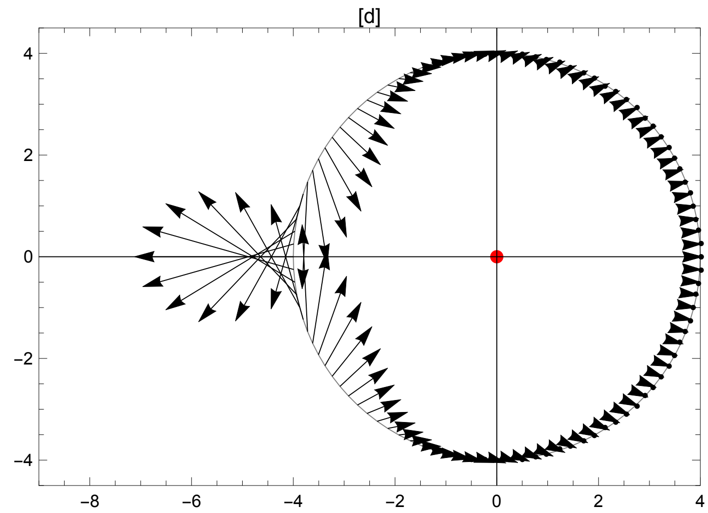


$(x-1)^4/(x+2)^7$

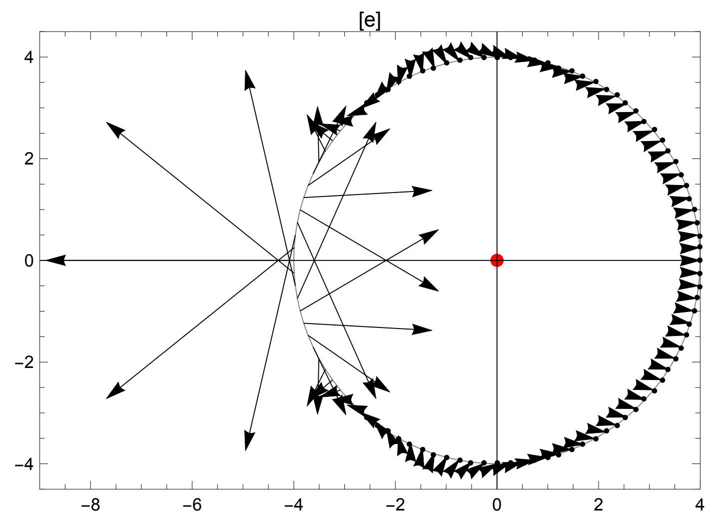
[c]



$1/x$



$(z-1)^2/(z+2)^3$



$(z-1)^4/(z+2)^7$

The discussion on pp. 458, bottom, suggests that the root appears as a parabola and the pole appears as a vertical asymptote and that both functions in x look something like 1/z, as in [c]. The plots of vector fields are explained with reference to winding and indexes. There is no indication of how one might infer the locations of roots and poles from the plots of vector fields. Nothing in [d] obviously reveals that the root is 1. In [e], one can see that the vectors expand rapidly suggesting that the pole is -2.