

Needham, Chapter 10, Example 8, p. 469.

G. Palmer, February 12, 2018, 9 pm, PT

8 If f and g are continuous, one-to-one mappings of the sphere S to itself, then their composition $f \circ g$ is too. Let us prove that at least one of these three mappings must possess a fixed point. We proceed by the method of contradiction.

(i) Show if none of the mappings possess a fixed point then, for each point p of S , the points p , $f(p)$ and $[f \circ g](p)$ must be distinct.

If none of the three mappings possess a fixed point, then

$$f(p) \neq p \quad f \text{ is distinct from } p \quad (1)$$

$$g(p) \neq p \quad g \text{ is distinct from } p \quad (2)$$

$$[f \circ g](p) \neq p \quad f \circ g \text{ is distinct from } p \quad (3)$$

$$[f \circ g](p) = f(g(p)) = f(\text{not } p) \implies f \circ g \text{ is distinct from } f, \text{ by (2)}$$

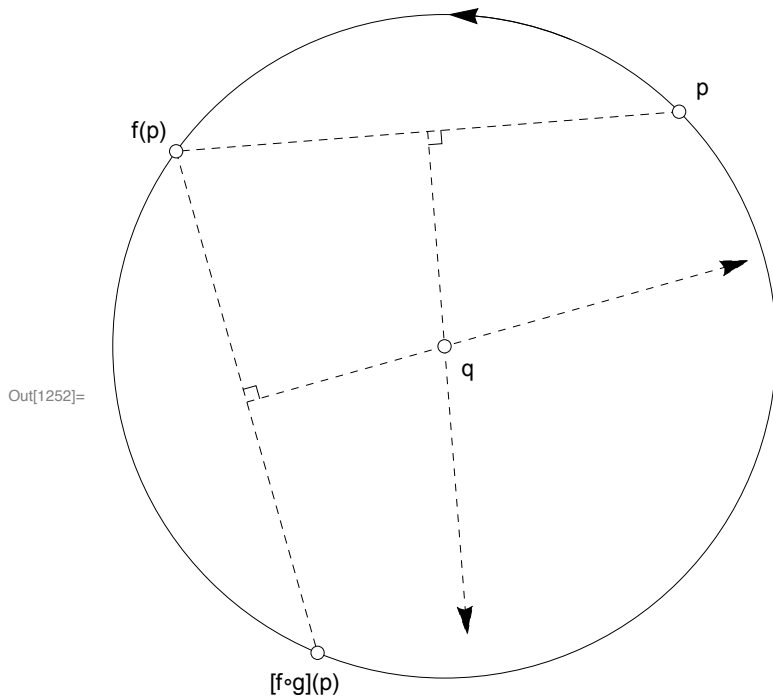


Figure 1. Directed circle on three points
in plane determined by three points on sphere

(ii) In this case, deduce that there is a unique, directed circle C_p on S passing through these three points in stated order.

Three points on a sphere determine a plane that intersects the sphere. A circle drawn through any three distinct points on a plane is unique. No matter where the three points occur, a circle C_p can be drawn passing through the points in stated order, i.e. $p \rightarrow f(p) \rightarrow [f \circ g](p)$ by directing the circle plot either anticlockwise as in (Figure 1) or clockwise. $g(p)$ can not be on C_p .

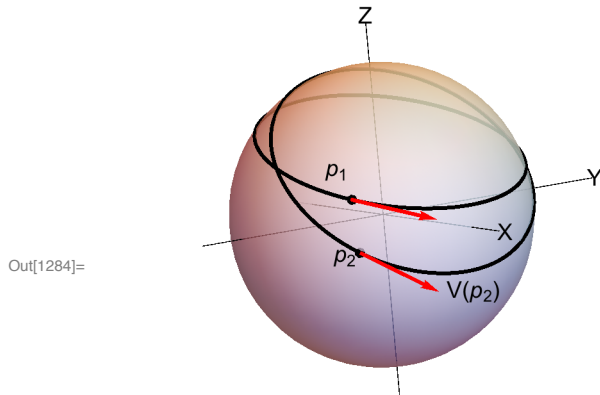


Figure 2. Two vectors of V field on C_p

(iii) Imagine a particle orbiting on C_p at unit speed, and let $V(p)$ be its velocity vector as it passes through p . Since p was arbitrary, V is a vector field on S .

(iv) By appealing to the Poincaré-Hopf Theorem, obtain the desired contradiction.

$V(p)$ is by definition a unit vector at every point on S , because the choice of p on S is arbitrary. $V(p)$ never vanishes or goes to infinity. Therefore it is never a singular point and there can never be a singular point on the sphere under this construction. By p. 464, ¶ 2, a vector field without any singular points can only exist on a topological doughnut. That is a contradiction, so one of the three mappings f , g and $f \circ g$ must possess a fixed point. Furthermore, even though the particle orbits on every C_p , only velocity vectors at $V(p)$ are considered (for all possible p) as belonging to the vector field, so two V can never emanate from the same point. This means that $\mathcal{I}_V[S]$ cannot be defined on any tangent plane of S , so it is undefined on S . This contradicts the Euler characteristic ($\chi = 2$) of the Poincaré-Hopf theorem applied to the sphere.