

Needham, Chapter 10, Exercise 7, p. 469.
G. Palmer, February 7, 2018, 3:20 pm, PT

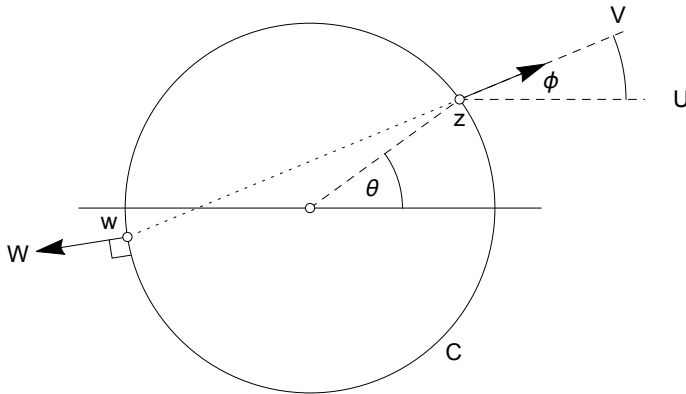


Figure 1. Prasolov's problem

7 Given a vector field V , defined on a circle C , let a vector field W be constructed on C in the manner illustrated If $\mathcal{I}_V[C] = n$, find $\mathcal{I}_W[C]$. [This problem is taken from Prasolov [1995, Chap. 6] and the answer may be found there too.]

There are two important facts here: (1) W is orthogonal to C and remains that way. (2) The chord is an extension of V to intersect C (phrasing is Vasco's). Therefore, as z traverses C once in the anti-clockwise direction, V rotates at a certain rate dictated by $\mathcal{I}_V[C] = n$, the chord rotates with V , and W rotates as dictated by the intersection of the chord and W on C . In the following, we refer to the traversal of z and w (end of chord opposite from z) on C and the rotations of V and W , each around its own tail or its tail translated to the origin. This rotation of a field vector about its tail is what is meant by $\mathcal{I}_V[C] = n$ and $\mathcal{I}_W[C]$. Since the chord is always parallel to V , the chord is tangent to a streamline of V at z (Figure 1). In Figure 1, a few features have been added to Needham's figure.

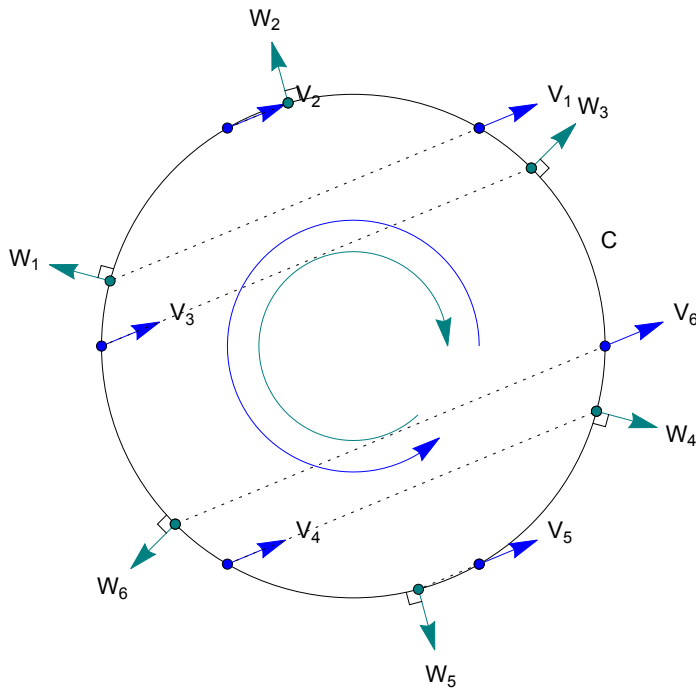


Figure 2. Prasolov's problem; case 1, fixed V

Answer 1, by geometric construction

Case 1. Establish a fiducial vector field U parallel to the real line so that V is a constant with $\phi = 0$ (unlike Figures 1 and 2, which show ϕ in a constant positive angle with respect to U). Then it is possible to see, using a pencil or some physical vector substitute) that the point w (at the opposite end of the chord from z) traverses clockwise on this field as z traverses C . Because W remains orthogonal to C , each full traversal of w induces a rotation of -2π in W . In this situation, $\mathcal{I}_V[C] = 0$ (V is constant), $\mathcal{I}_W[C] = -1$. Figure 2 illustrates a similar situation, except that the vector field is not parallel to the real line.

Case 2. Let C be a unit circle. Hold z constant (0 traversal), let $\phi = \arg(V)$, set $\phi_0 = 0$. Then $\arg(W) = \pi$ at the start. Advance ϕ to $\arg(\pi/2)$. The point w traverses from $e^{i\pi}$ to 1; advancing ϕ to π causes w to traverse from 1 to π , completing one full traversal. Since W must remain orthogonal to C , it completes a full rotation. Since V completed only a half rotation, it follows that for every full rotation (2π) of V , there are two rotations of W . If $\mathcal{I}_V[C] = n$, then $\mathcal{I}_W[C] = 2n$.

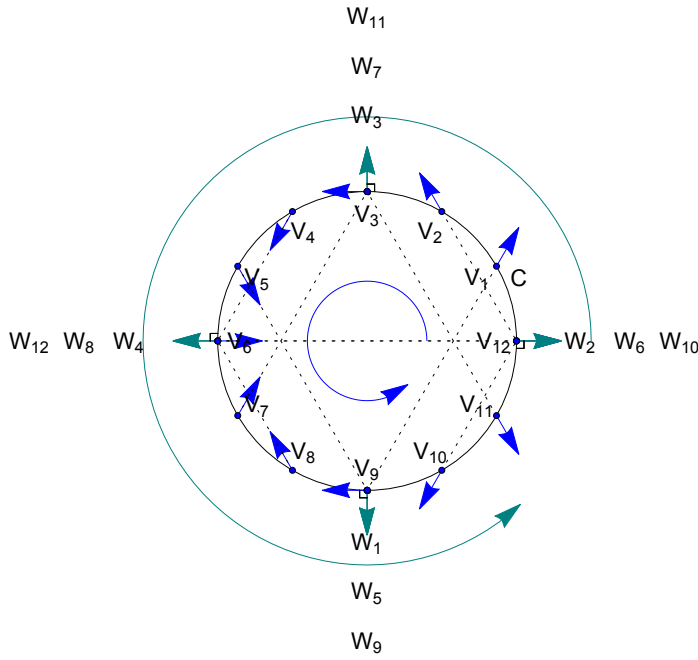


Figure 3. Prasolov's problem; $V = z^2$

To obtain a solution where V undergoes n rotations while z traverses C one time in the anticlockwise direction, add the indices from both cases:*

z traverses, V constant	$\mathcal{I}_{V1}(0) = 0$	$\mathcal{I}_{W1}(0) = -1$
V rotates, z constant	$\mathcal{I}_{V2}(0) = n$	$\mathcal{I}_{W2}[C] = 2n$
z traverses & V rotates	$\mathcal{I}_V(0) = n$	$\mathcal{I}_W[C] = 2n-1$

Notice that $\mathcal{I}_W[C] = 2n-1$ works for case 1, where $n = 0$. Figure 3 illustrates the case of $V = z^2$. We know from the double root that $n = 2$, so $\mathcal{I}_W[C] = 3$. In Figure 3, one can see from the stacks of labels for W that there are three winds. W_0 is not labeled, but it would be located at -1 .

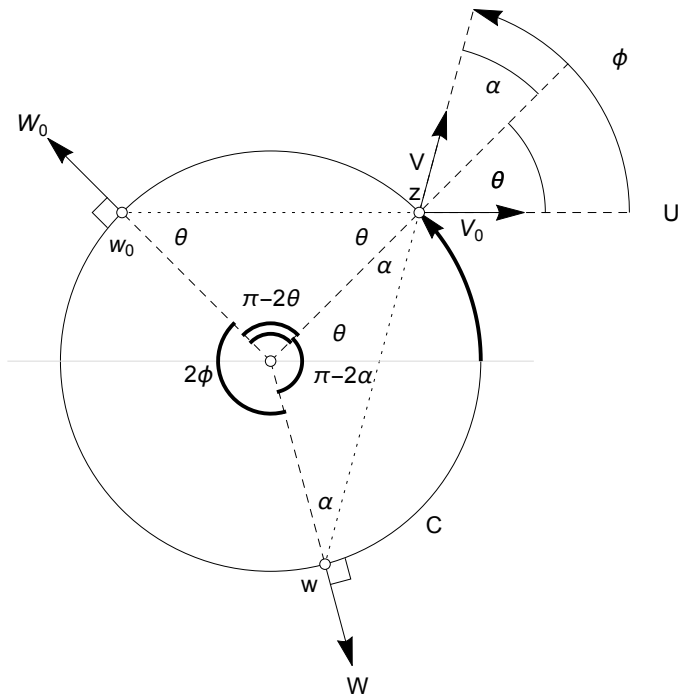


Figure 4. Finding index of W with complex algebra

Answer 2. By algebra based on geometric construction (Figure 4).

Let C be a unit circle centred at the origin. This should cause no loss of generality, because the radius of C has no effect on the angle of the traversal of w or the rotation of W . Let U be parallel to the real line.

Since we know that W is always orthogonal to C , the main objective is to trace and describe the traversals of w . Then we can infer the rotation of W . Assume that V remains parallel to the real line. Then

$$w_0 = e^{i(\theta + (\pi - 2\theta))} = e^{i(\pi - \theta)} = -e^{-i\theta}$$

We can see that wherever we place z_0 , if z rotates through an angle of $\theta = 2\pi$ about the origin, w rotates $-\theta = -2\pi$ about the origin traversing C once in the clockwise direction. $\mathcal{I}_W(0) = \nu[w, 0] = -1$.*

Now consider the rotation of V simultaneously with the traversal of z . Figure 4 shows that, since $\alpha = \phi - \theta$, whenever V rotates by an angle of ϕ about z transposed to the origin, w rotates by 2ϕ . Now we can write

$$w = w_0 e^{2\phi} = -e^{-i\theta} e^{2\phi}$$

Thus, a complete anticlockwise rotation of V by an angle ϕ about z transposed to the origin will send w around C twice in the anticlockwise direction, and $v[W, 0] = 2I_V[C] = 2n$. Now we can write

$$I_W[C] = 2I_V[C] + I_W(0) = 2n - 1.$$

QED

*We could also write $I_V[C] = n$ for $I_V(0) = n$, keeping in mind that V traverses C just once with z , but undergoes n rotations. The index notation seems inadequate to distinguish rotations of W associated with the traversal of w and those associated with the rotation of V . At Vasco's suggestion, we use the notation " $I(s)$ " suggested at the bottom of p. 457 to index rotations about a point, zero in this case, induced by the clockwise traversal of w on C . In this problem, there is only one circle on which to traverse, so $I_V[C]$ indicates the index of the vector V arising from traversal of C and $I_V(0)$ indicates windings of V around the point 0 .