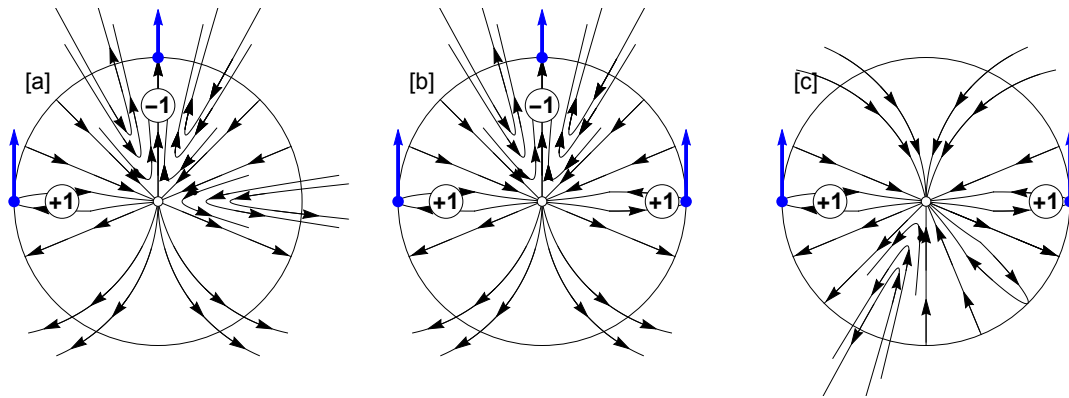


Needham, Exercises 5, 6, p. 468.

G. Palmer, January 23, 2018, 1:30 pm, PTs



5 Find the index of each of the three singular points shown below. [Same as above].

Following Poincaré's method, illustrated on p. 460, we plot a simple loop in such a way that we can determine places where the field vectors all point in the same direction. Then we determine the direction of rotation of the field vector from one side of a point to the other as z traverses a loop in the anticlockwise direction. Considering all points in the chosen direction, calculate $I = P - N$ (positive - negative rotation of the field vector at the selected places). Places where field vectors could be vertical are at the ends of the loops in the elliptic sectors on the real line and at northward pointing boundaries between sectors. We plot circles touching the ends of loops in elliptical sectors (See Example 6 for definition of elliptic sector). For the purpose of orientation, we can treat the three circles as unit circles. The field vectors are plotted in blue. Vector length is of no significance. As determining direction of rotation can be difficult by inspection alone, Vasco has suggested placing a pencil tangent to the streamline and observing the rotation of the pencil as it passes over a selected point. It is an useful method.

$$[a] \ P = 1, N = 1 \quad I = 1 - 1 = 0$$

$$[b] \ P = 2, N = 1 \quad I = 2 - 1 = 1$$

$$[c] \ P = 2, N = 0 \quad I = 2 - 0 = 2$$

In [a], there appears to be a streamline at i pointing north on the boundary between two hyperbolic sectors. The figure for Example 5, more so than the figure above, makes it appear that there is a negative rotation from one side of the point to the other.

In [c], we note the northward pointing streamline at $-i$. To the west of i , the streamline in example 5 [c] appears to have a negative rotation as z approaches i and a positive rotation as z moves away from i . We then deduce that the rotation at the point is zero.

6 Observe that the neighborhood of every singular point we have examined in this chapter is made up of sectors of one of the three types shown below, called elliptic, parabolic, and hyperbolic. Let e , p , and h denote the number of each type of sector surrounding a singular point.

(i) Verify that the index of each of the three singular points in the previous question is correctly (and painlessly) predicted by Bendixson's Formula:

$$[a] \mathcal{I} = 1 + \frac{1}{2}(e - h) = 1 + \frac{1}{2}(1 - 3) = 0$$

$$[b] \mathcal{I} = 1 + \frac{1}{2}(2 - 2) = 1$$

$$[c] \mathcal{I} = 1 + \frac{1}{2}(3 - 1) = 2$$

We verify that the indices of the singular points are predicted by Bendixson's Formula.

(ii) Explain this formula.

Bendixson's Formula omits the number of parabolic sectors and calculates the difference between the numbers of elliptic and hyperbolic sectors. May we deduce that every critical point where sectors adjoin and intersect has one and only one parabolic sector, which is represented in the formula as 1, and that the index is actually determined by the number of elliptic (+) and hyperbolic (-) sectors? It also appears that $(e-h)$ must be even, so e and h of 0 and 0, 3 and 1, 5 and 3, 4 and 2, or some such would work in the formula. If $(e-h)$ were odd, it would imply half-winds around the critical point. Half-winds are not counted.