

Needham, Chapter 10, Exercise 4, p. 468
G. Palmer, January 22, 2018, 9:10 am, PT

4 Suppose that one of the streamlines in a vector field is a simple closed loop L . Explain why L must contain a singular point of V .

Needham wrote on p. 453, *By definition, a vector field is everywhere tangent to the streamlines, and thus its direction can be recovered from the phase portrait.* (p. 453)

Proof by contradiction: Given a simple closed loop L that is a streamline, assume that L does not contain a singular point. Then the index $I(s) = I(\Gamma_s)$ must vanish (p. 458 (i)). Since the index is equivalent to a winding number, there is no winding number, so there is no wind (of V on L). Since L is a streamline there is traversal without a wind. Hence, L can not be a simple closed loop.

An argument could also be constructed from the fact that “there can be no vector field on the sphere that is free of singular points” (p. 462) using stereographic projection. One would project the simple closed loop and its critical point from the Riemann sphere to the complex plane.