

If q is a critical point, $R'(q) = 0$ and $R(q) = \text{const}$. Then q lies on a sphere of radius R which touches S at q , and pq is a ray orthogonal to S at q .

(ii) The level curves of R on S are the intersections of S with the “onion” of concentric spheres centred at p . Sketch these level curves in the vicinity of the illustrated critical points of R . Notice the distinction between points where R has a local maximum or minimum, versus points where R may increase or decrease depending on the direction (within S) in which one moves away from the critical point.

Figure 1 represents an attempt to illustrate local maxima (q_1, q_3), a local minimum (q_2), and a saddle near the critical point q_4 (“where R may increase or decrease depending on the direction (within S) in which one moves away from the critical point”). In the gravitational field of Earth, the local maxima become gravity sinks and the local minima become gravity sources.

(iii) Imagine that p generates an attractive force field, so that every point particle in space experiences a force F directed towards p . For example, we could imagine that p is the centre of the Earth, and that F is the Earth’s gravitational field. If a particle is constrained to move on S , then the only part of F to which it can respond is the projection F_S of F onto S . Sketch the streamlines of F_S . How are they related to the level curves of R in (ii)?

The streamlines are sketched very inaccurately as radial arrows, indicating only whether they flow towards or away from a critical point. The actual curvature of the intervening surface is not indicated. In the vicinity of critical points, the streamlines of F_S approach tangency to the level curves.

(iv) You have just seen that the critical points of R are the singular points of F_S . How does the index $\mathcal{I}(q)$ of a singular point of F_S distinguish between the types of critical point discussed in (ii).

For the singular points of local maxima and local minima, $\mathcal{I}(q) = +1$. It does not distinguish between sources and sinks. For the saddle points, $\mathcal{I}(q) = -1$.

(v) Let us define multiplicity of a normal pq to be this index $\mathcal{I}(q)$. Use the Poincaré-Hopf Theorem to deduce that

The total number of normals (counted with their multiplicities) that may be drawn to S from any point p is independent of both the location of p and the precise shape of S , and is equal to $\chi(S)$.

From (12) we know that a sphere or torus can be divided into polygons, each having a source at the centre and sinks at the vertices with saddle points on the edges. We know that an equation for the index is $\mathcal{I}(S) = F - E + V = 2 - 2g$. The Euler characteristic specified by the RHS is constant for each topological type of surface (sphere, torus, double torus, etc.). Since all the critical points of a surface are normals to a single arbitrary point not on the surface (but different for each such point), the total number of normals (counted with their multiplicities) to an exterior point must equal the Euler characteristic of S , and since these surfaces may be distorted into new shapes of the same

type, the number of normals that may be drawn from any point p is independent of both the location of p and the precise shape of S .

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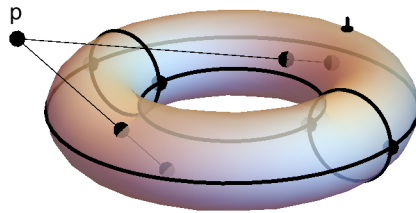


Figure 2. Flotation device with geodesics

(vi) Verify Reech's theorem for a couple of positions of p in the case where S is a torus (doughnut).

The flotation device in Figure 2 has been divided into four rectangles (on genus 1). The lower line from p is orthogonal to S and passes through the rectangle and the center of a meridian (not shown) to touch the lower rectangle, where it is also orthogonal to S . The same is true of the upper line from p to the other two rectangles. The 16 sides of the four rectangles have 8 edges. There are four vertices, two on each of the two meridians where they intersect the parallels.

For the torus, $g = 1$.

$$\text{Position 1. } p = \left\{ -3 + \frac{1}{\sqrt{2}} - 3\sqrt{2}, 0, -\frac{1}{\sqrt{2}} + 3\sqrt{2} \right\}$$

$$F = 4, E = 8, V = 4$$

$$4 - 8 + 4 = 2 - 2 \cdot g = 0$$

From this position we could rotate p around the z axis keeping z constant, or we would invert the torus by rotating it by Π on the x or y axis. So long as we don't place p on a direct line passing through the plane of the two meridians or the plane of the two parallels, and so long as all the lines from p to the torus are orthogonal, the result will be the same.