

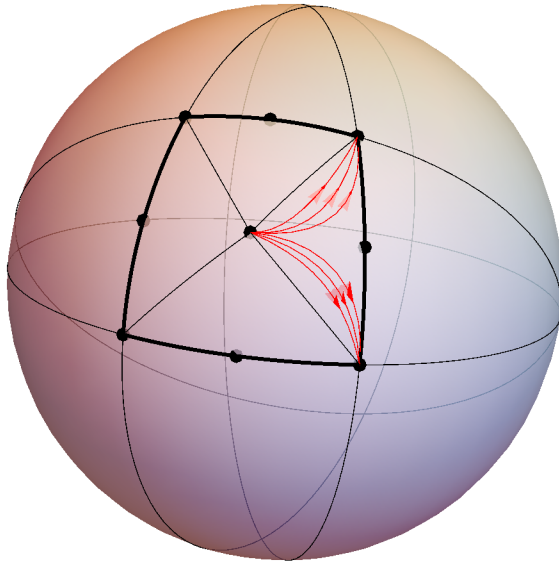


Figure 1. Vector fields, critical points,
and polygons on sphere

12 Let S be a smooth closed surface of genus g , so that its Euler characteristic is $\chi(S) = 2 - 2g$. As in [14], let us divide S into F polygons, and let E and V be the total number of edges and vertices, respectively.

(i) Draw a simple example on the surface of an orange and convince yourself (by drawing it) that we may obtain a consistent flow over the entire surface whose only singular points are (1) a source inside each of the F polygons; (2) a simple saddle point on each of the E edges; (3) a sink at each of the vertices.

From Figure 1 we deduce that whatever non-overlapping polygons we can draw to cover the sphere, we can triangulate any (convex) polygon by choosing a single source point in the interior of the polygon and drawing lines from this point to its vertices. Then streamlines, such as the red lines in Figure 1, can be drawn from the source to the sinks at the vertices. When all the polygons are triangulated and all the streamlines drawn, a saddle point appears at each of the edges. If “consistent flow” means that this pattern covers the sphere without exception or irregularity, then I am convinced.

(ii) By applying the Poincaré-Hopf Theorem to such a flow on the general surface S , deduce the following remarkable result, called Euler’s Formula:

$$F - E + V = \chi(S).$$

From Figure 1, we can deduce

- (a) The central point is a source that extends streamlines into every triangle that meets at this point. The number of central points is equal to F , the number of polygons. For a source, $\mathcal{I} = +1$ (pp. 457, 458).
- (b) the streamlines emanating from the source points of two contiguous triangles from two adjacent polygons form a simple saddle point. The number of saddle points is equal to E . For a saddle point, $\mathcal{I} = -1$.
- (c) every vertex is a sink for streamlines emanating from triangles in adjacent polygons that touch at the vertex. All the streamlines that disappear at a single vertex merge into single sink. The number of sinks equals V . For a sink, $\mathcal{I} = +1$.

Then from Exercise 11 (ii), for the sphere (or orange), for which $g = 0$, we have:

$$+ F - E + V = 2 = 2 - 2g = \chi(S)$$

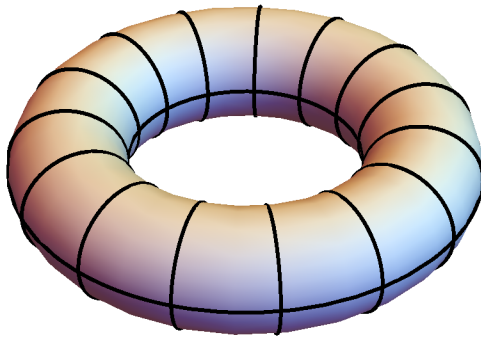


Figure 2. Torus with geodesics

(iii) *Verify this result for your example in (i),*

The example in Figure 1 has 8 triangles and 6 rectangles. $F = 14$, $E = 24$, and $V = 12$. Then $14 - 24 + 12 = 2$. QED

then try it out on a doughnut.

The theorem as presented so far appears to depend on polygons whose sides are geodesic lines. Figure 2 shows a torus (doughnut) with 15 meridians and 2 geodesic parallels. These furnish the sides of 30 rectangles, so $F = 30$. Other geodesics exist for the torus, but I am not familiar with their

geometry nor do I know whether one can cover the torus with polygons plotted on other geodesic lines. One can see that all of the rectangles lying between the two geodesic parallels on the top and bottom can be triangulated and streamlines added as was illustrated for the rectangle on the sphere in part (i). These 30 rectangles on the torus have 120 sides, which gives 60 edges, so $E = 60$. They have a total of 30 vertices, so $V = 30$. Then, with $g = 1$,

$$F - E + V = 30 - 60 + 30 = 0 = 2 - 2g$$

The Poincaré-Hopf Theorem and Euler's characteristic agree on both examples.