

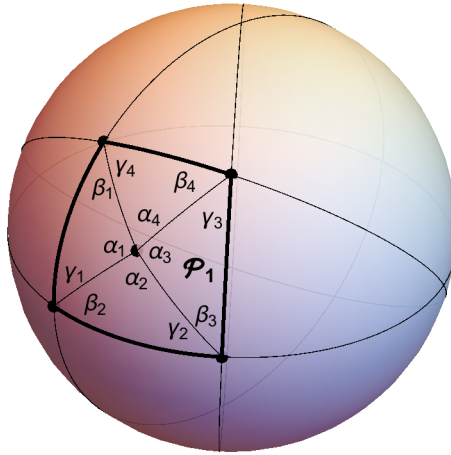


Figure 1. Polygon on sphere

11 Imagine the surface of the unit sphere divided up into F polygons, the edges all being “straight lines on the sphere”, i.e. great circles. Let E and V be the total number of edges and vertices that result from dividing up the sphere in this way.

(i) Let $\mathcal{P}_n$ be an n -gon on the unit sphere. Use (9), p. 279, to show that $\mathcal{A}(\mathcal{P}_n) = [\text{angle sum of } \mathcal{P}_n] - (n-2)\pi$.

In Figure 1, the facing polygon $\mathcal{P}_1$ has been divided into four triangles. One can see that it has four edges and four vertices. In this part, the number of edges and vertices will not be of concern. Instead, note that the angle sum for $\mathcal{P}_1$ is the sum of all the β and γ angles. The sum of all the α angles equals 2π . From p. 279 we have $\mathcal{E}(T) = k \mathcal{A}(T)$, where T is a triangle on the sphere and $k = \frac{1}{R^2}$. Since $R = 1$, $\mathcal{A}(T) = (\alpha + \beta + \gamma - \pi)$. The shapes of the polygons and the locations of the critical points within the triangles are left unspecified, so we can't assume that corresponding angles of triangles are equal.

Then for any polygon with “straight” edges on the unit sphere,

$$\begin{aligned} \mathcal{A}(T_i) &= (\alpha_i + \beta_i + \gamma_i - \pi) \\ &= (\beta_i + \gamma_i) + \alpha_i - \pi \end{aligned}$$

Since the sum of the central interior angles $\sum_{i=1}^n \alpha_i$ equals 2π ,

$$\begin{aligned}\mathcal{A}(\mathcal{P}_n) &= \sum_{i=1}^n (\beta_i + \gamma_i) + 2\pi - n\pi \\ &= [\text{angle sum of } \mathcal{P}_n] - (n-2)\pi\end{aligned}$$

QED

(ii) By summing over all the polygons, deduce that

$$F - E + V = 2.$$

[This argument is due to Legendre (1794); the result itself is a special case of the result in the following exercise.]

Sum the areas of all the $\mathcal{P}_n$ to obtain the area of the sphere (4π):

$$\mathcal{A}(S) = \sum_F [\text{angle sum of } \mathcal{P}_n] - \sum_{i=1}^F (n_i \pi) + 2\pi F = 4\pi \quad (1)$$

The sphere is covered by contiguous and non-overlapping $\mathcal{P}_n$, where n may be any positive integer. At each vertex, the sum of the angles between the edges is 2π . The sum of the angles at all the vertices is therefore equivalent to the angle sum of all the $\mathcal{P}_n$.

$$\sum_F [\text{angle sum of } \mathcal{P}_n] = 2\pi V \quad (2)$$

Since n is the number of sides of a polygon and it takes two congruent sides of neighboring $\mathcal{P}_n$ to make one edge, $E = (\sum_{i=1}^F n_i)/2$.

$$\sum_{i=1}^F (n_i \pi) = 2\pi E \quad (3)$$

Then rewrite (1) substituting with (2) and (3) and dividing through by 2π .

$$2\pi F - 2\pi E + 2\pi V = 4\pi$$

$$F - E + V = 2$$

QED

