

Needham, Chapter 10, Exercise 1, p.468.

G. Palmer, November 1, 2018, 12:20 pm PDT

1 Show both algebraically and geometrically that the streamlines of the vector field z^2 are circles that are tangent to the real axis at the origin. Explain why the same must be true of the vector field $1/\bar{z}$.

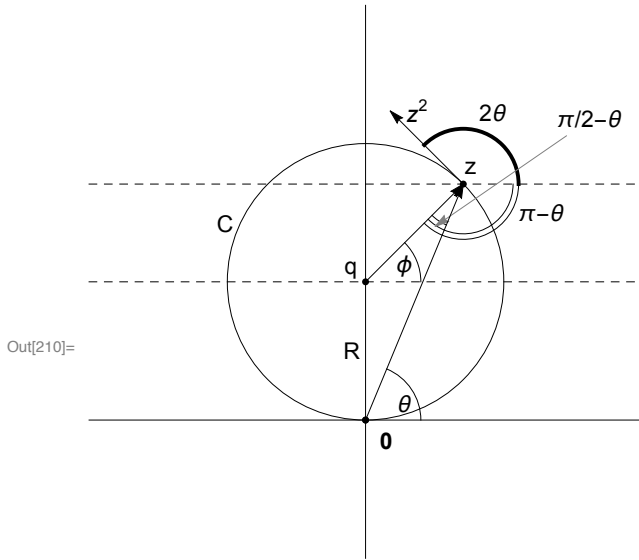


Figure 1. Vector field z^2

Let $z = re^{i\theta}$. If the streamlines of z^2 are circles tangent to the real axis at the origin, then z^2 must at every point be tangent to one of the circles. We will deal only with circles in the UHP. A circle tangent to the real axis at the origin can be described by the formula $z = (R e^{i\phi} + iR)$, where R is the radius of a circle centred at $q = iR$. If z^2 is tangent to the circle C at all points on C , then z^2 is orthogonal to the spoke $(z-q) = R e^{i\phi}$ for all z .

Geometric proof

In Figure 1, one can see that z^2 is orthogonal to $(z-q)$ if the partial circle around z has a total angle of $3\pi/2$. Starting with the angle of z^2 and proceeding clockwise, one can see that the angle sum can be composed as $2\theta + (\pi-\theta) + (\pi/2-\theta) = 3\pi/2$. Done.

Algebraic proof

If $z^2 \perp (z-q)$ then $z^2 \cdot (z-q) = 0$. We have to show that $(R e^{i\phi} + iR)^2 \cdot R e^{i\phi} = 0$. Since the length of the vectors is irrelevant, let $R = 1$. Then we only need to show that $(e^{i\phi} + i)^2 \cdot e^{i\phi} = 0$.

$$(e^{i\phi} + i)^2 \cdot e^{i\phi} = (e^{i2\phi} + 2ie^{i\phi} - 1) \cdot e^{i\phi}$$

$$\begin{aligned}
&= (\cos 2\phi + i \sin 2\phi + 2i(\cos \phi + i \sin \phi) - 1) \cdot (\cos \phi + i \sin \phi) \\
&= (\cos 2\phi + i \sin 2\phi + 2i \cos \phi - 2 \sin \phi - 1) \cdot (\cos \phi + i \sin \phi) \\
&\equiv \begin{pmatrix} \cos 2\phi - 2 \sin \phi - 1 \\ \sin 2\phi + 2 \cos \phi \end{pmatrix} \cdot \begin{pmatrix} \cos \phi \\ \sin \phi \end{pmatrix} \\
&= (\cos 2\phi - 2 \sin \phi - 1) \cos \phi + (\sin 2\phi + 2 \cos \phi) \sin \phi \\
&= \cos \phi \cos 2\phi - 2 \cos \phi \sin \phi - \cos \phi + \sin \phi \sin 2\phi + 2 \cos \phi \sin \phi \\
&= \cos \phi \cos 2\phi + \sin \phi \sin 2\phi - \cos \phi \\
&= \cos \phi (\cos^2 \phi - \sin^2 \phi) + \sin \phi (2 \sin \phi \cos \phi) - \cos \phi \\
&= \cos \phi \cos^2 \phi - \cos \phi \sin^2 \phi + 2 \cos \phi \sin^2 \phi - \cos \phi \\
&= \cos \phi \cos^2 \phi + \cos \phi \sin^2 \phi - \cos \phi \\
&= \cos \phi (\cos^2 \phi + \sin^2 \phi) - \cos \phi \\
&= 0
\end{aligned}$$

QED

Explain why the same must be true of the vector field $1/\bar{z}^2$.

Let $z = re^{i\theta}$. Then

$$\frac{1}{\bar{z}^2} = \frac{1}{r^2 e^{-i2\theta}} = \frac{e^{i2\theta}}{r^2}$$

The vector field $1/\bar{z}^2$ has the same direction (2θ) at every point as z^2 , so it must be tangent to a circle of radius r at every point on the circle, which must therefore be a streamline.