Needham, Chapter 1, Exercise 9, p. 46.
G. Palmer, January 1, 2020, 12:05 pm PST

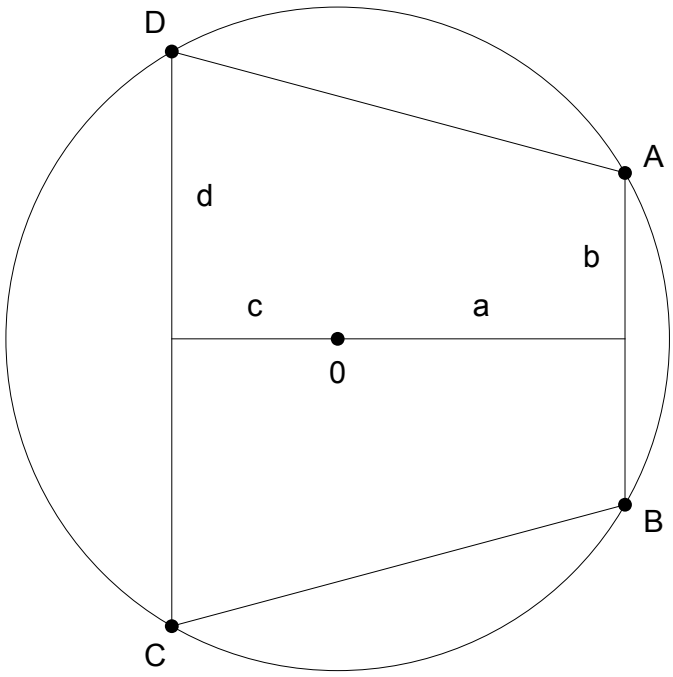


Figure 1. A+B+C+D ≠ 0

9 Let A, B, C, and D be four points on the unit circle. If A+B+C+D = 0, show that the points must form a rectangle.

Since Needham specified "**the** unit circle", we can assume it is centred at the origin. Suppose four complex points A, B, C and D are symmetrical about the real axis but not about the y axis. Let a, b, c, and d be positive real numbers where A = a+bi, B = a-bi, C = -c-di, and D = -c+di, as in Figure 1. Adding just the real parts, we get (2a -2c), which is generally not zero. Adding just the imaginary parts, we get (b - b + d - d) = 0. Then we can say that generally A+B+C+D≠0. We deduce that if A+B+C+D=0, the four points must be arranged symmetrically about both axes.

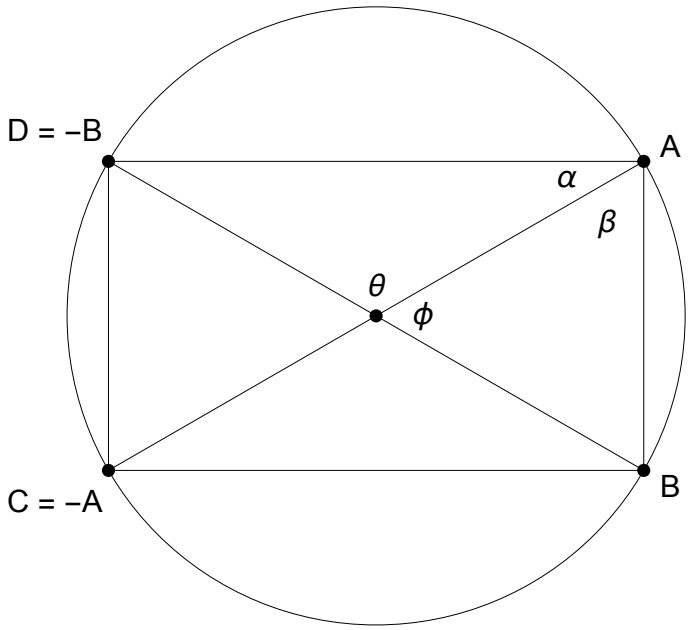


Figure 2. A+B+C+D = 0 is a rectangle

If A, B, C, and D are arranged symmetrically about both axes, one can plot diameters of opposing end points through the origin. On the unit circle, opposing vertices are equal and opposite in sign, so A+B+C+D = 0, and lines connecting opposing pairs cross at 0. In figure 2, it must be the case that C = -A and D = -B, so that A + C = 0 and B + D = 0. Then

$$\theta+\phi = \pi$$

$$2\alpha = \pi-\theta$$
$$2\beta = \pi-\phi$$

Add the left and right sides.

$$2(\alpha+\beta) = 2\pi -(\theta+\phi)$$

Obtain the angle at vertex A by substituting $\theta+\phi = \pi$.

$$(\alpha+\beta) = \pi - (\theta+\phi)/2 = \pi/2$$

Then by the symmetry of the opposing angles of crossed lines and the symmetry of the circle, the inside angles at all four vertices equal $\pi/2$. This is one definition of a rectangle. The four points may be rotated about the centre point by any angle, so long as it is the same for all, and the resulting rectangle is congruent. The result may be translated to a circle with a random centre point q, where A + B + C + D = q.