Needham, Chapter 1, Exercise 8, p. 46.
G. Palmer, December 30, 2019, 11:30 am PST

Out[6389]=

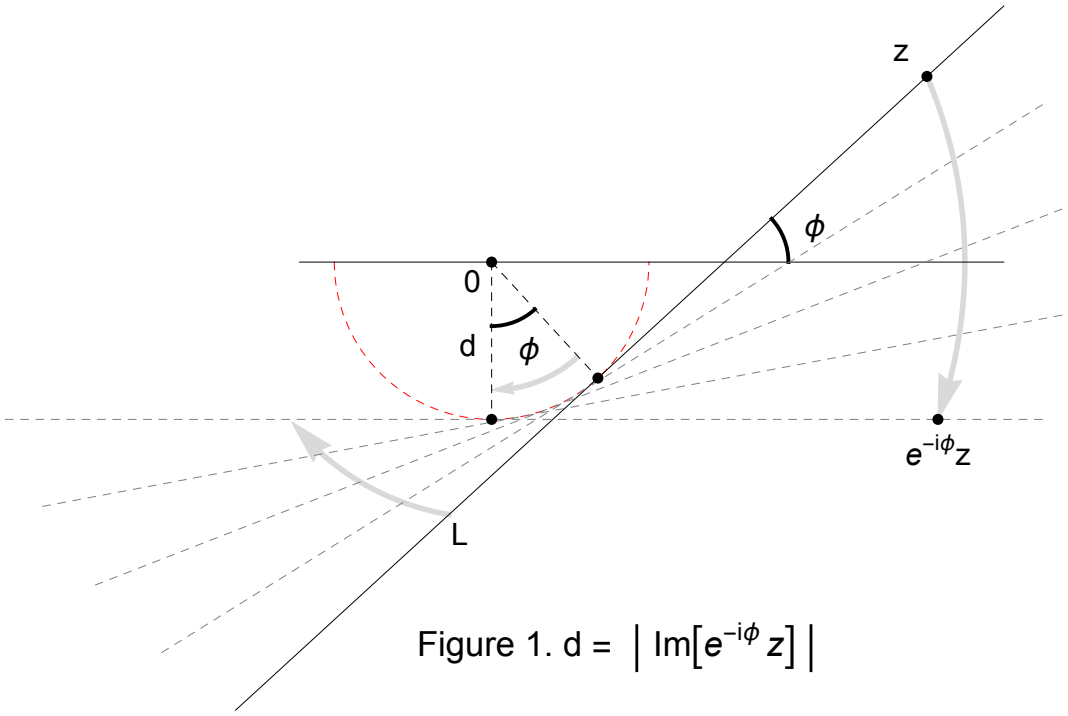


Figure 1. d = $\left| \text{Im}\left[e^{-i\phi}\, z\right] \right|$

8 Let L be a straight line in ℂ making an angle ϕ with the real axis, and let d be its distance from the origin. Show geometrically that if z is any point on L then

$$d = \left| Im[e^{-i\phi}\, z] \right| .$$

[Hint: Interpret $e^{-i\phi}$ *using (9).]* (p. 8)

We can see that $e^{-i\phi}$ can be interpreted as an expansion by a factor of R = 1 (no expansion) and a clockwise rotation of ∠(-ϕ) about the origin applied to all of ℂ, so that if z = $re^{i\theta}$ its angle after applying the rotation $e^{-i\phi}$ is ∠(θ-ϕ). The key point that I failed to grasp on the first try is that $e^{-i\phi}$ rotates the whole of ℂ. Consider what it does to L as shown in Figure 1. As L is rotated in the clockwise direction (gray arrows), the tangent point with the circle of radius d rotates by -ϕ, bringing L into a horizontal orientation lying on the line y = $-\left| \text{Im}\left[e^{-i\phi}\, z\right] \right|$. The distance d from the origin remains constant.

Out[6424]=

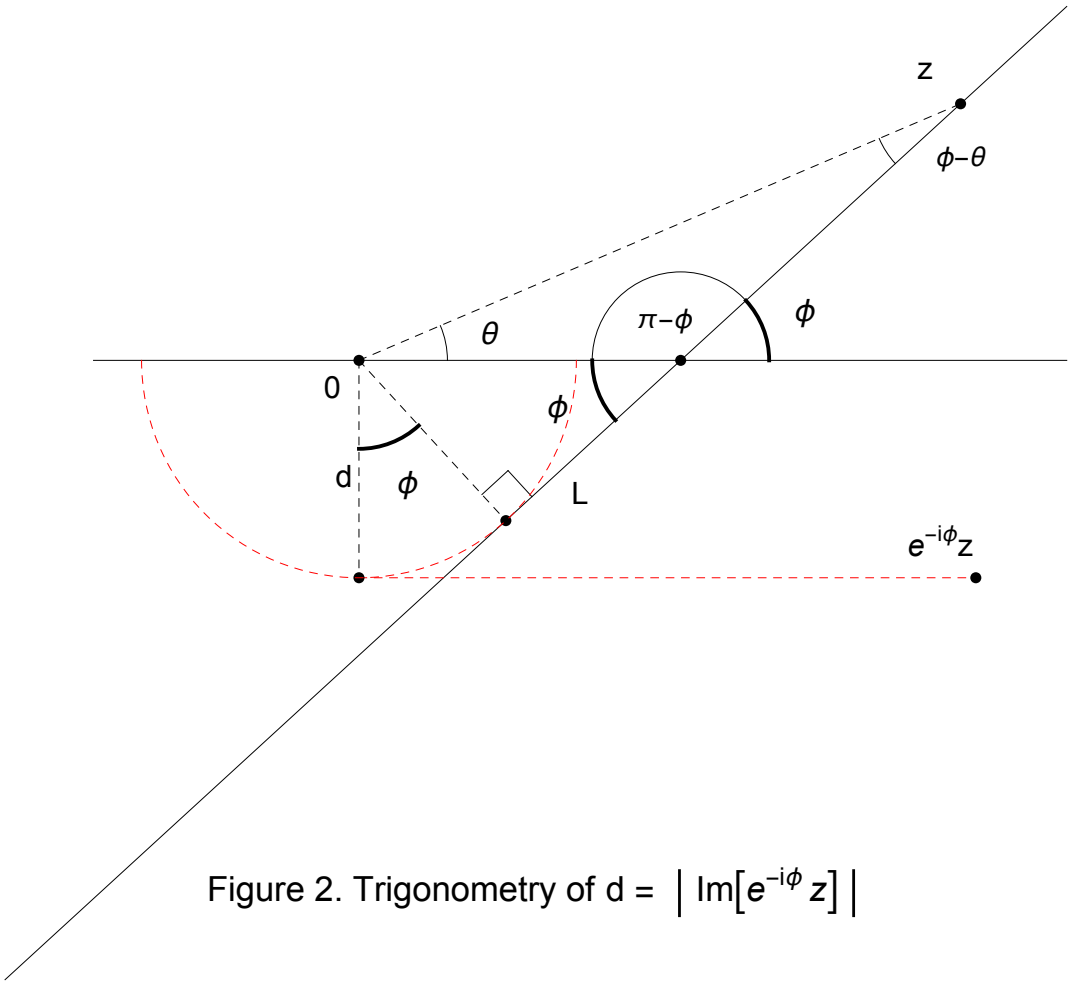


Figure 2. Trigonometry of d = $\left| \text{Im}\left[e^{-i\phi}\, z\right] \right|$

We can also prove the formula for d with algebra. It is evident that d = |x sin[ϕ]|, but to find d from z, we can show that d = |r sin(ϕ-θ)| as in Figure 2, and that this is equivalent to d = $\left| \text{Im}\left[e^{-i\phi}\, z\right] \right|$. Note that $e^{-i\phi}\, z$ is not the same as $\bar{z}$. The horizontal red dashed line is y = -d = $\text{Im}\left[e^{-i\phi}\, z\right]$. The superimposed red figure of circle and horizontal line illustrates the fact that d = $\left| \text{Im}\left[e^{-i\phi}\, z\right] \right|$.

$$d = \left| \text{Im}\left[e^{-i\phi}\, z\right] \right| = \left| \text{Im}\left[e^{-i\phi}\, r e^{i\theta}\right] \right|$$

$$= |\text{Im}[r\,(\cos(\theta-\phi) + i\,\sin(\theta-\phi))]| = |r\,\sin(\phi-\theta)|$$