Needham, Chapter 1, Exercise 45, pp. 53-54.
G. Palmer, May 17, 2020, 12:45 pm PST

45 In the previous exercise you showed that an arbitrary dilative rotation or translation can be written as a complex function of the form f(z) = az +b, and conversely, that every such function represents a unique dilative rotation or translation.

(i) Given two pairs of distinct points {A, B} and {A′, B′}, show [by finding them explicitly] that a and b exist such that f(A) = A′ and f(B) = B′.

By the problem statement, $\{A', B'\}$ can be written as {a A + b, a B + b}.

Then

$$a = \frac{(A' - b)}{A} \qquad \text{and} \qquad b = (B' - a\,B)$$

$$aA = A' - b$$

$$b = A' - aA = B' - a\,B$$

$$aB - aA = B' - A'$$

$$a = \frac{B' - A'}{B - A}$$

$$b = B' - \frac{B' - A'}{B - A}B = \frac{A'\,B - AB'}{B - A}$$

(ii) Deduce the result (31)

We need a proof that f preserves proportions of line segments because *A similarity S is a mapping of the plane to itself that preserves ratios of distances* (p. 40). In order to determine whether f preserves the proportionality of line segments, we introduce a third point and write

$$A' = aA + b$$
$$B' = aB + b$$
$$C' = aC + b$$

We test the proposition that

$$\frac{B' - A'}{B - A} = \frac{C' - A'}{C - A}$$

$$\frac{(aB + b) - (aA + b)}{B - A} = \frac{(aC + b) - (aA + b)}{C - A}$$

$$(B - A)(aC + b - aA - b) = (C - A)(aB + b - aA - b)$$

$$\begin{aligned} &\mathrm{BC}\,a + \mathrm{BR}b - \mathrm{ABa} - \mathrm{BRb} - \mathrm{ACa} - \mathrm{ARb} + A^2\,a + \mathrm{A}b \\ &\quad = \mathrm{CaB} + \mathrm{C}b - \mathrm{CAre}^{\mathrm{i}\alpha} - \mathrm{C}b - \mathrm{AB}a - \mathrm{AR}b + A^2\,a + \mathrm{A}b \end{aligned}$$

$$0 = 0$$

The equation balances, showing that the proportions of line segments are preserved.

Then we can write $f(z) = \mathcal{D}_0^{r,\alpha}(z) + v$ showing that f(z) is a composition of the dilation r and the motions rotation by α about 0 and translation by v. The composition of two direct similarities is another direct similarity.

Let $r = |a|$, $R = |b|$, $\alpha = \arg(a)$, $\beta = \arg(b)$. Then $a = re^{i\alpha}$, $b = Re^{i\beta}$

$$f(z) = re^{i\alpha}\,z + Re^{i\beta} = \mathcal{D}_0^{r,\alpha}(z) + Re^{i\beta}$$

In Exercise 44, it was established that

$$\mathcal{T}_v \circ \mathcal{D}_0^{r,\theta} = \mathcal{D}_p^{r,\theta},\ \textit{where}\ p = v \,/\, \left(1 - re^{i\theta}\right)$$

Then

$$f(z) = az + b = \mathcal{T}_b \circ \mathcal{D}_0^{r,\alpha}(z)$$

$$= \mathcal{D}_p^{r,\alpha}(z),\ \textit{where}\ p = b \,/\, \left(1 - re^{i\alpha}\right)$$

It follows that *Every direct similarity is a dilative rotation or (exceptionally) a translation*.