Needham, Chapter 1, Exercise 43, pp. 53-54.
G. Palmer, May 16, 2020, 11:15 am PST

43 On p. 18 we calculated that

$$\mathcal{T}_v \circ \mathcal{R}_0^{\alpha} = \mathcal{R}_c^{\alpha}, \text{ where } c = v \Big/ \left(1 - e^{i\alpha}\right)$$

(i) Show algebraically that the complex number from the old centre of rotation (the origin) to the new centre of rotation (c) has length $\frac{|v|}{2\sin(\alpha/2)}$*, and that its direction makes an angle of* $\left(\frac{\pi-\alpha}{2}\right)$ *with v.*

By considering Figure [13b], one can deduce that rotation and translation require parallel lines with internal half angles that sum to π, so the angles sum to 2π. The centre of rotation c is analogous to (b – *a*) in exercise 42. The ray c makes an angle of $\left(\frac{\pi-\alpha}{2}\right)$ with v. The only algebra required is to substitute c for (b – *a) and α for θ.* But we can show the work.

The distance from the origin to c is $|c| = \left| v \Big/ \left(1 - e^{i\alpha}\right) \right|$.

$$\left(1 - e^{i\alpha}\right) = (e^{i0} - e^{i\alpha}) = -i\ 2\ \sin\left(\frac{\alpha}{2}\right) e^{i\frac{\alpha}{2}} \quad \text{(by Ex. 18 \& Ex. 42)}$$

so

$$|c| = |v| \Big/ \left| 1 - e^{i\alpha} \right| = \frac{|v|}{2\sin(\alpha/2)}$$

and the direction of c is

$$\arg(v \Big/ \left(1 - e^{i\alpha}\right)) = \arg(v) - \arg\left(1 - e^{i\alpha}\right).$$

Since

$$\left(1 - e^{i\alpha}\right) = -i\ 2\ \sin\left(\frac{\alpha}{2}\right) e^{i\frac{\alpha}{2}},$$

it has the direction of $\arg(-ie^{i\frac{\alpha}{2}}) = \arg\left(e^{i\left(\frac{\alpha}{2}-\pi/2\right)}\right)$, so

$$\arg(c) = \arg(v) - \left(\frac{\alpha}{2} - \pi/2\right) = \arg(v) + \left(\frac{\pi-\alpha}{2}\right),$$

which means that c makes an angle of $\left(\frac{\pi-\alpha}{2}\right)$ with v.

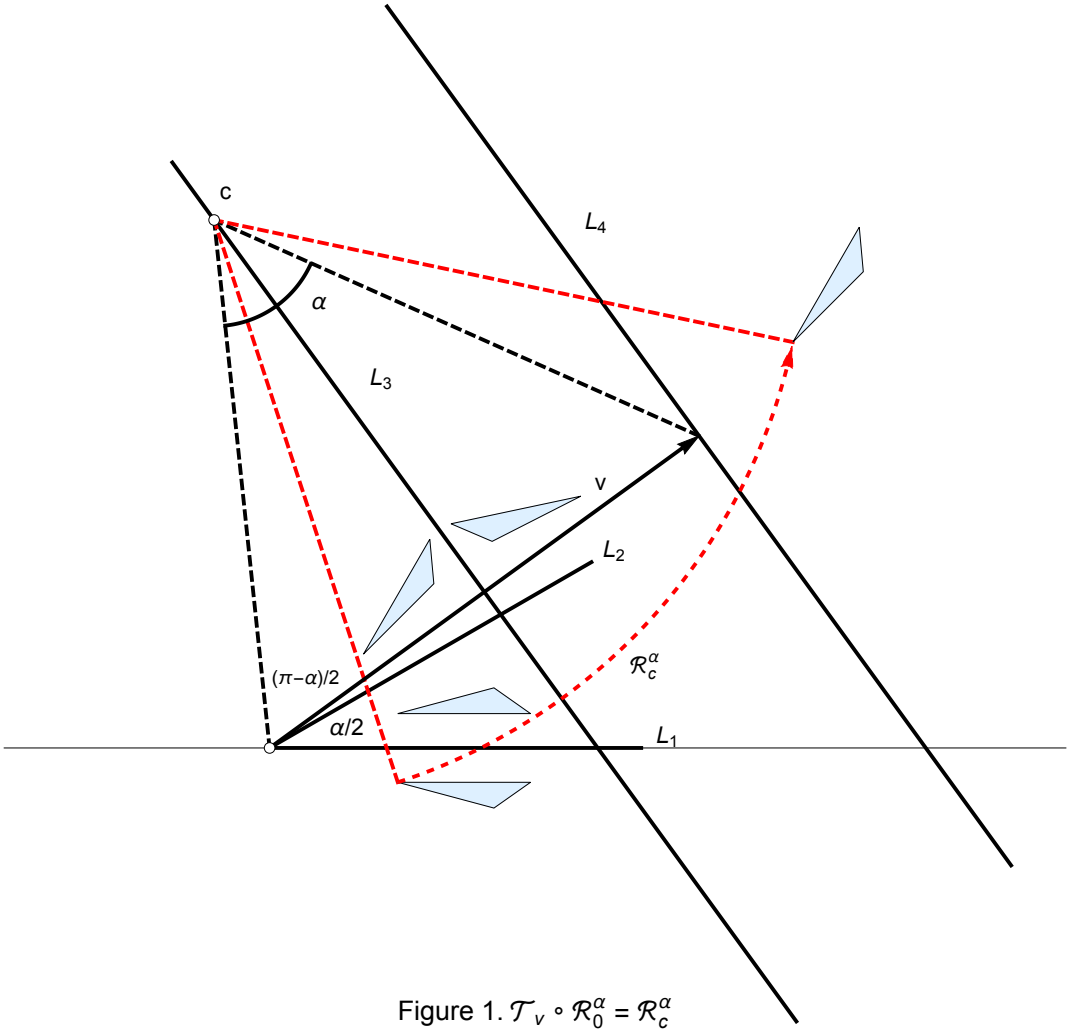


Figure 1. $\mathcal{T}_v \circ \mathcal{R}_0^\alpha = \mathcal{R}_c^\alpha$

(ii) Representing both $\mathcal{R}_0^\alpha$ and $\mathcal{T}_v$ as the composition of two reflections, use the idea in [30b] to give direct, geometric proofs of these results.

The essential idea from [30b] is that two lines of reflection (from two rotations each centred at its own point) can be merged to reveal that the two rotations may be displayed as a single rotation from a third point. It is implicit in the equation on line -3 of p. 38, that reflection about the merged line is the identity reflection. Hence, we find only the first and last reflections written explicitly on the RHS. The identity reflection represented by the second and third reflections has become tacit and in fact unnecessary. What Vasco has done in his answer (Forum) is to merge the first line of reflection *from the translation* with the second line of reflection *from the rotation* about 0. In effecting the merger, one cannot bring the line from the rotation to the translation, because the rotation is anchored at 0. Similarly, one cannot rotate the line of reflection from the translation to agree with a line of reflection from the rotation, because that would skew the translation vector v, which we take to be arbitrary and given. The reflection vectors of the translation must remain perpendicular to v. Therefore, one must rotate both the reflection lines (keeping the angle between them constant)

associated with $\mathcal{R}_0^\alpha$ and translate both the reflection lines associated with $\mathcal{T}_v$ (keeping the distance between them constant) in the direction of v until the first line of the translation merges with the second line of the rotation. Note that the rotations and translations required to merge two lines of reflection are not the same as those specified in the problem statement, i.e., they are not v and α.

Figure 1 shows the rotation $\mathcal{R}_0^a$ is accomplished in two reflections, first on L_1 and then on L_2. It also shows that the translation $\mathcal{T}_v$ is accomplished in two reflections on lines L_3 and L_4, which are orthogonal to v. The angle at c between the dotted lines is α, showing that the result is $\mathcal{R}_c^\alpha$. Since L_3 is the perpendicular bisector of v and the dotted line from c to the tip of v is a perfect rotation of $\angle\alpha$, the triangle with apex at c and base v is isosceles, so the angle between c and v must be $(\pi\text{-}\alpha)/2$. The sine rule says that

$$(|v|/2)/\sin(\alpha/2) = |c|/\sin(\text{Pi}/2)$$

so

$$|c| = \frac{|v|}{2\sin(\alpha/2)}$$

This verifies the results from part (i).

The red dashed lines show that the rotation $\mathcal{R}_c^\alpha$ matches the left hand corner of the triangle after it has been rotated and translated. Note that the angle between L_1 and L_2 is $\alpha/2$, which is required to obtain R_0^α by the double reflection. It is not necessary that v be parallel to L_1. The translation vector v is arbitrary.

Out[4555]=

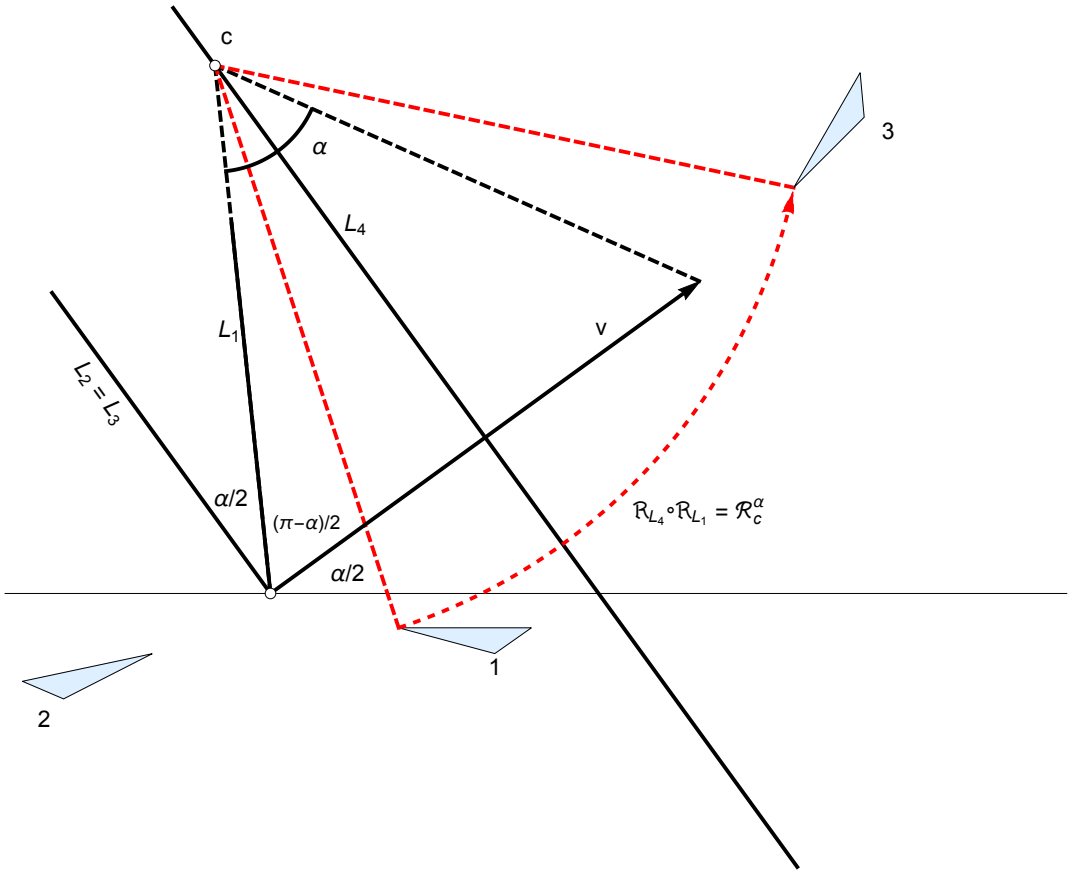


Figure 2. $\mathcal{T}_v \circ \mathcal{R}_0^\alpha = \mathcal{R}_c^\alpha$ with merged lines of reflection

Figure 2, shows what happens when lines L_1 and L_2 are rotated so that L_2 is parallel to L_3 and L_4, and then L_3 and L_4 are translated on v until L_3 merges with L_2. Then $\mathcal{R}_c^\alpha$ requires only $\mathfrak{R}_{L_4} \circ \mathfrak{R}_{L_1}$. The result is the same as if L_3 had not been merged with L_2. The calculation is made with only two reflections instead of four. Since the number of reflections is even, the sense of triangle 1 is preserved in 3.