Needham, Chapter 1, Exercise 42, pp. 53.
G. Palmer, May 10, 2020, 9:05 am PST

42 On p. 19 we calculated that if $(\theta + \phi) = 2\pi$ then

$$\mathcal{R}_b^{\phi} \circ \mathcal{R}_a^{\theta} = \mathcal{T}_v, \text{ where } v = \left(1 - e^{i\phi}\right)(b - a)$$

(i) Let Q = (b – a) be the complex number from the first centre of rotation to the second. Show algebraically that v has length 2 sin(θ/2)|Q|, and that its direction makes an angle of $\left(\frac{\pi-\theta}{2}\right)$ with Q.

$$\left(1 - e^{i\phi}\right) = (e^{i0} - e^{i\phi}) = \text{i } 2 \sin\left(\frac{0-\phi}{2}\right) e^{i\,\frac{0+\phi}{2}} \qquad \text{(by Ex. 18)}$$

Then substitute for v and (b – a).

$$v = i\,2 \sin\left(\frac{-\phi}{2}\right) e^{i\,\frac{\phi}{2}}\, Q \qquad (1)$$

Since $\theta + \phi = 2\pi$, $\sin(\frac{\theta}{2}) = \sin(\frac{\phi}{2})$.

$$|v| = \left| i\,2 \sin\left(\frac{\theta}{2}\right) e^{i\,\frac{\phi}{2}} \right| |Q|$$

$$= 2 \sin\left(\frac{\theta}{2}\right) \left| Q \right|$$

which shows that v has length $2 \sin(\theta/2)|Q|$.

We still need to show that $(\arg(v) - \arg(Q)) = \left(\frac{\pi-\theta}{2}\right)$. It is given that $v = \left(1 - e^{i\phi}\right)Q$ and $(\theta+\phi) = 2\pi$. Then

$$\arg(v) - \arg(Q) = \arg(v/Q) = \arg\left(1 - e^{i\phi}\right)$$

From (1) one can calculate that

$$\arg\left(1 - e^{i\phi}\right) = \arg\left(-i\,2 \sin\left(\frac{\phi}{2}\right) e^{i\,\frac{\phi}{2}}\right)$$

$$= \arg\left(e^{i\left(\frac{2\pi-\theta}{2} - \frac{\pi}{2}\right)}\right)$$

$$= \frac{\pi-\theta}{2}$$

This is what we wanted to show.

Out[1069]=

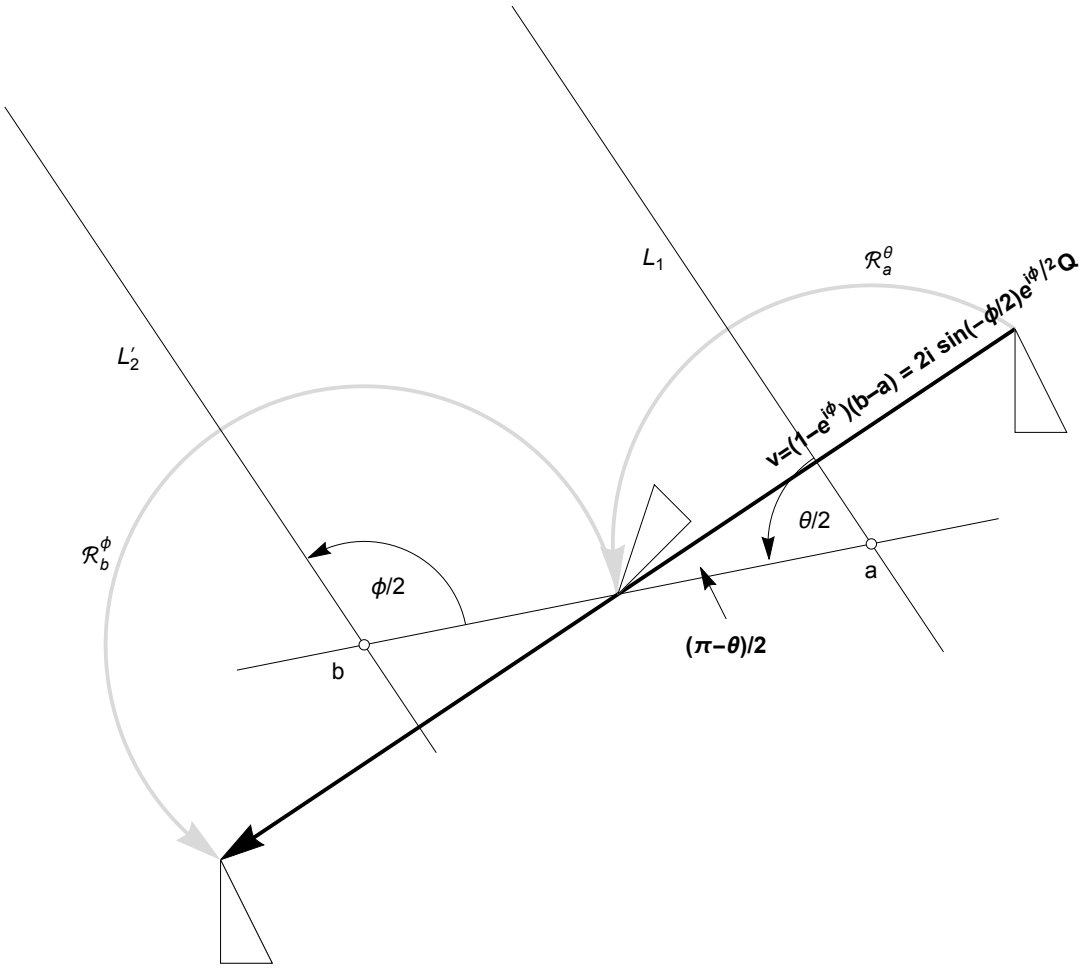


Figure 1. Facsimile of [30b] with $\theta+\phi = 2\pi$

(ii) Give direct geometric proofs of these results by redrawing figure [30b] in the case $(\theta + \phi) = 2\pi$.

Figure 1 shows that the triangle on the lower left is the result of applying the rotations $\mathcal{R}_b^\phi \circ \mathcal{R}_a^\theta$ to the triangle on the upper right. One can see that all three triangles have the same sense or orientation. The first triangle on the upper right can be translated directly to the triangle on the lower left with a line of length (2 sin (ϕ/2) |Q|). Line L_1 becomes the perpendicular bisector of the line segment between the endpoints of $\mathcal{R}_a^\theta$, which entails that the acute angle complementary to θ/2 equals (π-θ)/2. This constitutes the geometric proof requested.