Needham, Chapter 1, Exercise 40, pp. 53.
G. Palmer, May 3, 2020, 12 noon PST

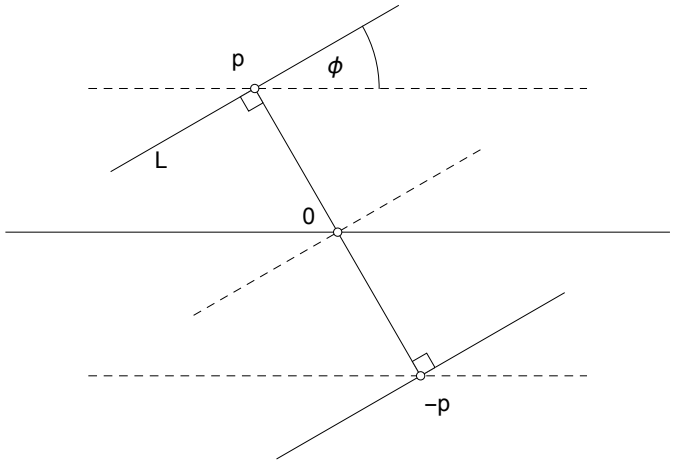


Figure 1. $p = \pm|p|e^{i\phi}$

40 Let L be a line making angle ϕ (or $\phi + \pi$) with the real axis, and let p be the point on L that is closest to the origin, so that |p| is the distance to the line. Consider the glide reflection [cf. previous exercise] $G = \mathcal{T}_v \circ \mathcal{R}_L$, where the translation is through distance r parallel to L. Let us fix the value of ϕ by writing $v = +re^{i\phi}$.

(i) Use a picture to show that $p = \pm \left| p \right| e^{i\phi}$ and explain the geometric significance of the ±.

See Figure 1. Every line segment L has a reflection $\tilde{L}$ in a parallel line through the origin. The closest point to the origin on the reflection is the negative of the closest point on the segment, and $\tilde{p} = -p$.

(ii) What transformation is represented by the complex function $H(z) = \bar{z} + r$?

This is a glide reflection with reflection $\mathcal{R}_X$ in the x-axis plus a translation $\mathcal{T}_r$ parallel to the x-axis. $H(z) = \mathcal{T}_r \circ \mathcal{R}_X$.

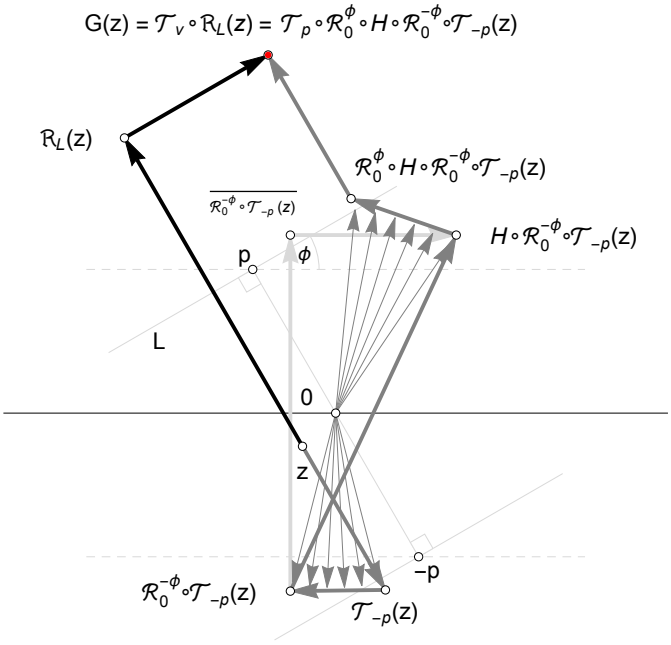


Figure 2. Two paths to G

(iii) Use pictures to explain why $G = \mathcal{T}_p \circ \mathcal{R}_0^{\phi} \circ H \circ \mathcal{R}_0^{-\phi} \circ \mathcal{T}_{-p}$.

The RHS of the equation is the glide reflection H applied to a composite function, which is a rotation (about the origin) applied to a translation. The result is given as the argument to a subsequent composite function, which is a translation applied to a rotation about the origin. The second composite function is the inverse of the first. The LHS is introduced as $G = \mathcal{T}_v \circ \mathfrak{R}_L$, which is a reflection followed by a translation, e.g. a glide reflection. So on the LHS, we have a glide reflection (Figure 2, black arrows). On the RHS we have a different glide reflection wrapped in inverse translations and rotations (Figure 2, gray arrows). Both movements arrive at the same point.

(iv) Deduce that $G(z) = e^{i2\phi}\,\bar{z} + e^{i\phi}(r \pm 2\,i\,|\,p\,|)$.

We are given that $G(z) = \mathcal{T}_v \circ \mathfrak{R}_L$. Rotate the complex plane about the origin by $-\phi$ to make L parallel to the x–axis. Then add –i|p| to the plane to make L congruent with the x–axis. The position of z has now become $e^{-i\phi}\,z$ – i|p|. Now reflect z across the x-axis (and L) to the point

$$\mathfrak{R}_{L=X} \circ \mathcal{R}_0^{-\phi}(z) = \overline{e^{-i\phi}\,z - i\,|\,p\,|} = e^{i\phi}\bar{z} + i|p|$$

Now translate the plane (and horizontal L) back by adding i|p| and rotate the plane back with $e^{i\phi}$, which sends L to its original orientation and z to its new position reflected in L.

$$\mathfrak{R}_L(z) = ((e^{i\phi}\bar{z} + i|p|) + i|p|)e^{i\phi} = e^{2\,i\phi}\bar{z} + 2\,i\,|\,p\,|\,e^{i\phi}$$

Apply the translation

$$\mathcal{T}_v \circ \mathfrak{R}_L = e^{2\,i\phi}\bar{z} + 2\,i\,|\,p\,|\,e^{i\phi} + \mathrm{re}^{i\phi}$$

$$= e^{2\,i\phi}\bar{z} + e^{i\phi}\ (\mathrm{r}+2\mathrm{i}|\mathrm{p}|)$$

Now include –2i|p| for the case in which L passes through (– p).

$$\mathrm{G(z)} = e^{2\,i\phi}\bar{z} + e^{i\phi}\ (\mathrm{r}\pm 2\mathrm{i}|\mathrm{p}|)$$

QED

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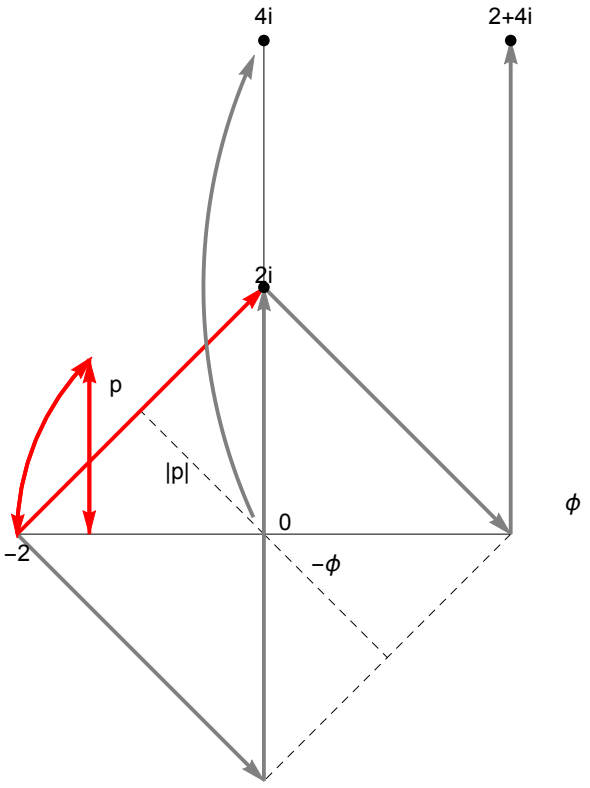


Figure 3. G(z) = i$\bar{z}$ + 4i

(v) Hence describe (in geometric terms) the glide reflection represented by G(z) = i$\bar{z}$ + 4i. Check your answer by looking at the images of -2, 2i, and 0.

G is a reflection about the real axis rotated by i = $e^{i\pi/2}$ and translated by 4i. But in the framework previously developed, it requires a line segment L at angle ϕ, a segment p representing the distance from the origin to the closest point on L. Then we achieved the reflection across L by rotating the plane by $-\phi$, translating the plane by ±i |p|, reflecting z across the real axis, inverting the translation and rotation, and applying v = $re^{i\phi}$, which is a movement parallel to L. We first calculate those values, given that G(z) = i$\bar{z}$ + 4i.

From (iv) we have

$$G(z) = e^{2\,i\phi}\bar{z} + e^{i\phi}\ (r \pm 2i|p|)$$

so we can write

$$e^{2\,i\phi}\bar{z} = i\bar{z} \qquad \text{and} \qquad 4i = e^{i\phi}\ (r \pm 2i|p|)$$

Then

$$e^{2\,i\phi} = i$$

$$2\phi = \pi/2 \qquad \Longrightarrow \qquad \boldsymbol{\phi = \pi/4}$$

From the first example,

$$4i = e^{i\pi/4}(r \pm 2i|p|)$$

$$4\ e^{i\pi/2} = e^{i\pi/4}(r \pm 2i|p|)$$

$$(r \pm 2i|p|) = 4\ e^{i\pi/4} = (2\sqrt{2} + 2\sqrt{2}\ i)$$

It follows that $\mathbf{r = 2\sqrt{2}\ ,\ |p| = \sqrt{2}\ ,\ p = \pm\, i\ \sqrt{2}\ e^{i\pi/4}}$, $\mathbf{v = 2\sqrt{2}\ e^{i\pi/4}}$ and

$$G(z) = e^{i\pi/2}\ \bar{z} + e^{i\pi/4}(2/\sqrt{2}\ \pm 2i\sqrt{2}\ )$$

$$= i\bar{z} + 4i$$

Applying the process described in the first paragraph, we obtain a pathway by which move from -2 to its image 2i (red arrows). But our new calculated value of G(z) = iz + 4i allows for more simple description. See the gray arrows for each of the examples.

The reflection of -2 is the identity transformation. Then rotate and add 4i.

$$-2 \Longrightarrow -2i + 4i = 2i$$

2i is reflected to -2i and rotated to 2. Then add 4i.

$$2i \Longrightarrow 2 + 4i$$

The reflection and rotation of 0 is the identity transformation. Add 4i.

$$0 \Longrightarrow 0 + 4i = 4i$$