Needham, Chapter 1, Exercise 39, pp. 52, 53.
G. Palmer, April 27, 2020, 12 noon PST

Out[5958]=

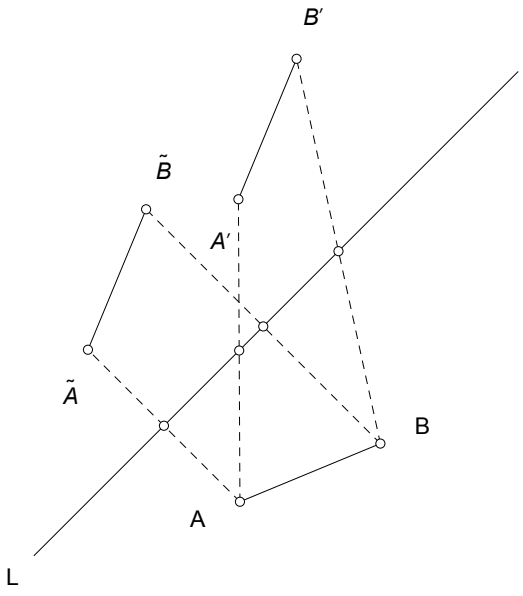


Figure 1. Glide reflection on L

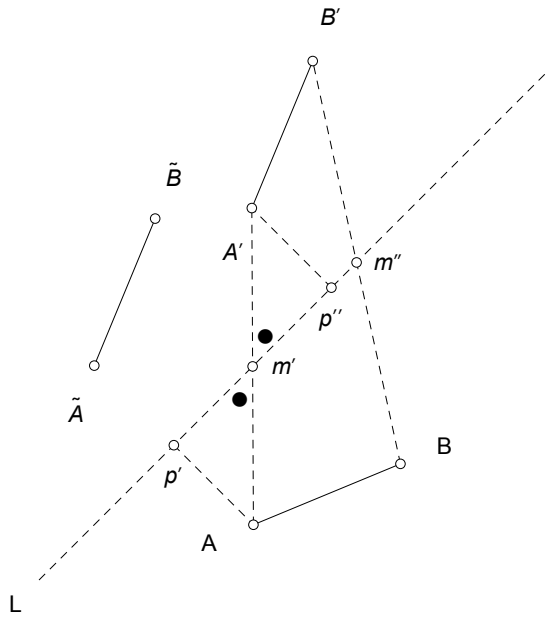


Figure 2. Reconstruct L

In[5443]:=

39 A glide reflection is the composition $\mathcal{T}_v \circ \mathcal{R}_L = \mathcal{R}_L \circ \mathcal{T}_v$ of reflection in a line L and a translation v in the direction of L....

(i) Draw a line L, a line-segment A B, the image $\tilde{A}\tilde{B}$ of the segment under $\mathcal{R}_L$, and the image A′ B′ of AB under the glide reflection $\mathcal{T}_v \circ \mathcal{R}_L$.

See Figure 1. Notice that L the lines AA and BB are both perpendicular to L, so they are parallel. By construction, L passes through their midpoints. Notice that L also passes through the midpoints of AA′ and BB′. It will be useful to prove this in part (ii).

(ii) Suppose you erased L from your picture; by considering the line-segments A A′ and B B′, show that you can reconstruct L.

One can deduce that m' and m'' in Figure 2 are midpoints from the fact that linear transformations preserve proportions of line segments. Since L bifurcates AA and BB, it must also bifurcate AA′ and BB′ (Figure 1).

Considering just the line-segments AA' and BB', find the midpoints m' and m'' and plot a line segment L through them (Figure 2). We prove that m' and m'' are midpoints by plotting projections p' and p'' onto L, showing that Ap′ ⊥ L and A' p'' ⊥ L. We know that $|\text{Ap}'| = |A'\ p''|$ by construc-

tion. One can see that $\angle Am' p' = \angle A' m' p''$. Since we have two angles and a side of two right triangles are equal, the triangles must be similar, confirming that m' is a midpoint. The same logic applies to m'' and confirms the reconstruction of L. It also reveals that the translation of $\tilde{A}\tilde{B}$ equals p''-p'.

(iii) Given any two segments A B and A' B' of equal length, use the previous part to construct the glide reflection that maps the former to the latter.

Reconstruct L from the midpoints of AA′ and BB′. Reflect AB across L to $\tilde{A}\tilde{B}$ and apply a translation to $\tilde{A}\tilde{B}$ in the same direction as L until $\tilde{A}\tilde{B}$ merges with $A'B'$.

$$A'B' = \mathcal{T}_v \circ \mathfrak{R}_L(\mathrm{AB}), \quad \text{where v} = p''\text{-}p'$$

(iv) Deduce that every opposite motion is a glide reflection.

In (iii) it was shown that any two line segments of equal length could be created with a glide reflection. The reflection always reverses the orientation of the angle between the line of reflection and a reflected segment. The glide conserves the orientation. Therefore, since an opposite motion is a motion that reverses orientation, every opposite motion is a glide reflection.

Out[5991]=

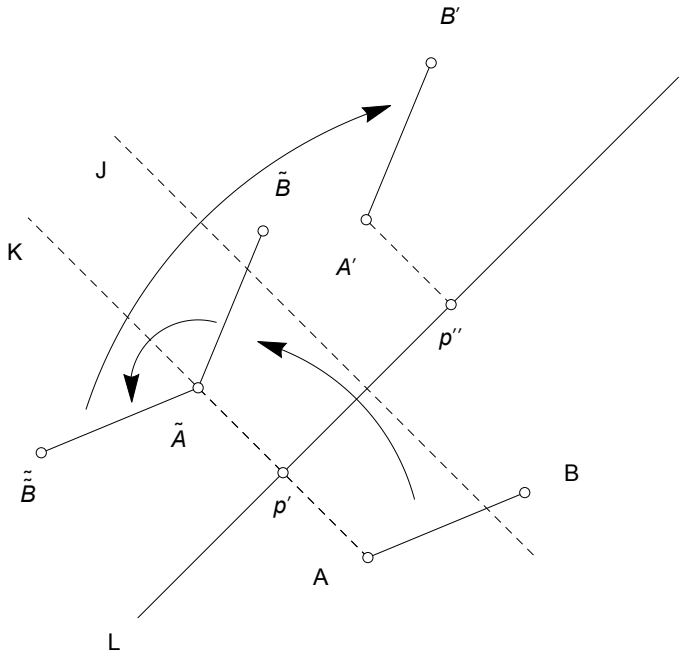


Figure 3. Glide reflection as composition of three reflections

(v) Express a glide reflection as the composition of three reflections.

Let K be the line segment A$\tilde{A}$ extended, which is perpendicular to L. Let J be the line segment parallel to K through the point $(1/2)(p'' - p') + p'$. Then $A'B' = \mathfrak{R}_J \circ \mathfrak{R}_K \circ \mathfrak{R}_L(\mathrm{AB})$.