Needham, Chapter 1, Exercise 38, p. 52.
G. Palmer, April 25, 2020, 9 am PST

38 On page 18 we calculated that for any non-zero α, $\mathcal{T}_v \circ \mathcal{R}_0^\alpha$ is a rotation.

$$\mathcal{T}_v \circ \mathcal{R}_0^\alpha = \mathcal{R}_c^\alpha, \qquad \text{where } c = v/(1 - e^{i\alpha}).$$

However, if $\alpha = 0$, then $\mathcal{T}_v \circ \mathcal{R}_0^\alpha = \mathcal{T}_v$ is a translation. Try to reconcile these facts by considering the behavior of $\mathcal{R}_c^\alpha$ in the limit that α tends to zero.

$$\mathcal{T}_v \circ \mathcal{R}_0^\alpha(\mathrm{z}) = z\, e^{\mathrm{i}\alpha} + \mathrm{v}$$

$$\mathcal{R}_c^\alpha = (\mathrm{z} - \mathrm{c})\, e^{\mathrm{i}\alpha} + \mathrm{c} = (\mathrm{z} - v/(1 - e^{i\alpha}))\, e^{\mathrm{i}\alpha} + v/(1 - e^{i\alpha})$$

We can see that c blows up as α approaches 0, so $\mathcal{R}_c^0$ is undefined. We need to return to the original equation $\mathcal{T}_v \circ \mathcal{R}_0^\alpha(\mathrm{z}) = \mathrm{z}\, e^{\mathrm{i}\alpha} + \mathrm{v}$.

$$\mathrm{z} \lim_{\alpha \to 0} (e^{\mathrm{i}\alpha}) + \mathrm{v} = \mathrm{z} \cdot 1 + \mathrm{v}, \text{ which is a translation.}$$