Needham, Chapter 1, Exercise 36, p. 52.
G. Palmer, April 23, 2020, 3 pm PST

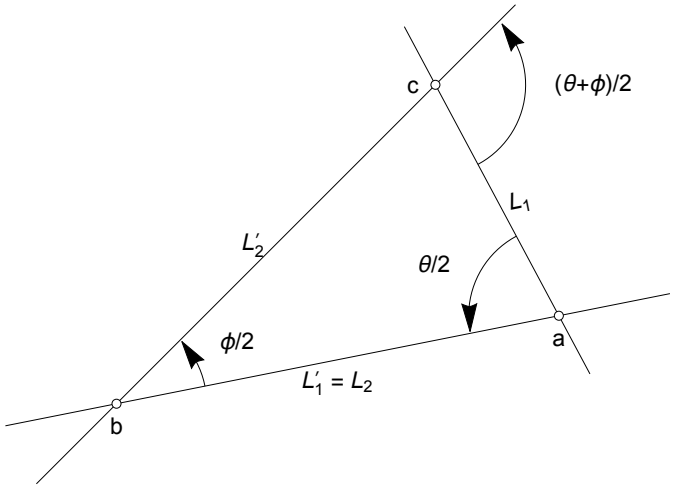


Figure 1. Facsimile of [30b]

36 According to the calculation on p. 18, $\mathcal{R}_b^{\phi} \circ \mathcal{R}_a^{\theta} = \mathcal{R}_c^{(\theta+\phi)}$*, where*

$$c = \frac{ae^{i\phi}\left(1-e^{i\theta}\right)+b\left(1-e^{i\phi}\right)}{1-e^{i(\theta+\phi)}}$$

Let us check that this c is the same as the one given by the geometric construction in [30b].

(i) Explain why the geometric construction is equivalent to saying that c satisfies the two conditions

$$arg\left[\frac{c-b}{a-b}\right] = \frac{1}{2}\,\phi \quad \text{and} \quad arg\left[\frac{c-a}{b-a}\right] = -\frac{1}{2}\,\theta$$

The angle abc, which is found by subtracting the angle of (a-b) from that of (c-b), can also be found as the argument of (c-b) divided by (a-b). It is named $\frac{1}{2}\,\phi$ because it is half the angle formed by the double reflection of z across (a-b) = $L_1' = L_2$ and then (c-b) = L_2' . The sign is positive because (c-b) is the larger positive angle in this construction.

Likewise, the angle cab, which was found by subtracting the angle of (b-a) from that of (c-a), can also be found as the argument of (c-a) divided by (b-a). It is named $-\frac{1}{2}\,\theta$ because it is half the angle formed by the double reflection of z across (c-a) = $L_1$and then (b-a) = $L_1' = L_2$. The sign is negative because (b-a) is the larger angle in this construction.

The geometric construction is equivalent to saying that c satisfies the two conditions because the location of c is determined by the two angles together with the line segment (b-a).

There is an apparent complication in the problem statement, because the second condition satisfied by c is a negative angle, but the angle is shown as positive (anti-clockwise rotation) in the diagram.

To my mind, it would be more natural to write the equivalent condition

$$\arg\left[\frac{b-a}{c-a}\right] = \frac{1}{2}\,\theta,$$

where arg(b-a) is a positive angle less than 2π.

(ii) Verify that the calculated value of c (given above) satisfies the first of these conditions by showing that

$$\frac{c-b}{a-b} = \left[\frac{\sin\frac{\theta}{2}}{\sin\frac{\theta+\phi}{2}}\right]e^{i\phi/2}$$

[Hint: Use $\left(1-e^{i\alpha}\right) = -2i\,\sin(\alpha/2)\,e^{i\alpha/2}$*.]*

I found it easier to work backwards on this problem. I show that the proper substitutions can be used to solve the above equation for c and arrive at the equation given in (i).

$$\text{c-b} = \left[\frac{\sin\frac{\theta}{2}}{\sin\frac{\theta+\phi}{2}}\right]e^{\text{i}\phi/2}\ (\text{a-b})$$

$$= \left[\frac{a\sin\frac{\theta}{2} - b\sin\frac{\theta}{2}}{\sin\frac{\theta+\phi}{2}}\right]e^{\text{i}\phi/2}$$

$$\text{c} = \frac{\left(a\sin\frac{\theta}{2} - b\sin\frac{\theta}{2}\right)e^{\text{i}\phi/2}}{\sin\frac{\theta+\phi}{2}} + \text{b}$$

Multiply numerator and denominator by $e^{\text{i}\theta/2}\,e^{\text{i}\phi/2}$. Multiply b by $\frac{1-e^{i(\theta+\phi)}}{1-e^{i(\theta+\phi)}}$.

$$= \frac{-2\,i\left(a\sin\frac{\theta}{2} - b\sin\frac{\theta}{2}\right)e^{\text{i}\theta/2}\,e^{\text{i}\phi}}{-2\,i\sin\frac{\theta+\phi}{2}\,e^{\text{i}\phi/2}\,e^{\text{i}\theta/2}} + \frac{b\left(1-e^{i(\theta+\phi)}\right)}{1-e^{i(\theta+\phi)}}$$

$$= \frac{\left(-2\,i\,a\sin\frac{\theta}{2}\,e^{\text{i}\theta/2} - \left(-2\,i\,b\sin\frac{\theta}{2}\,e^{\text{i}\theta/2}\right)e^{\text{i}\phi} + b\left(1-e^{i(\theta+\phi)}\right)\right)}{-2\,i\sin\frac{\theta+\phi}{2}\,e^{\text{i}\phi/2}\,e^{\text{i}\theta/2}}$$

Use $\left(1-e^{\text{i}\alpha}\right) = -2\text{i}\,\sin(\alpha/2)\,e^{\text{i}\alpha/2}$.

$$= \frac{\text{a}e^{\text{i}\phi}\left(1-e^{\text{i}\theta}\right) - \text{b}e^{\text{i}\phi}\left(1-e^{\text{i}\theta}\right) + b\left(1-e^{i(\theta+\phi)}\right)}{1-e^{i(\theta+\phi)}}$$

$$= \frac{a\,e^{\text{i}\phi}\left(1-e^{\text{i}\theta}\right) - \text{b}e^{\text{i}\phi} + \text{b}e^{i(\theta+\phi)} + b - \text{b}e^{i(\theta+\phi)}\right)}{1-e^{i(\theta+\phi)}} \quad \text{(collect and add like terms and factor)}$$

$$c = \frac{a e^{i\phi}\left(1-e^{i\theta}\right)+b\left(1-e^{i\phi}\right)}{1-e^{i(\theta+\phi)}}$$

In the same way, verify that the second condition is also satisfied.

The second condition: $arg\left[\frac{c-a}{b-a}\right] = -\frac{1}{2}\theta$

We can rewrite $\frac{c-a}{b-a}$ in the polar format:

$$\frac{c-a}{b-a} = \left|\frac{c-a}{b-a}\right| e^{-i\theta/2}$$

The length of segment (b-a) hinges on the angle at c.

$$\angle \text{bca} = \pi - (\theta+\phi)/2 = (2\pi-(\theta+\phi))/2 = -(\theta+\phi)/2$$

Using the sine rule, we can write:

$$\frac{\sin(-\phi/2)}{|c-a|} = \frac{\sin(-(\theta+\phi)/2)}{|b-a|}$$

$$\left|\frac{c-a}{b-a}\right| = \frac{\sin(\phi/2)}{\sin((\theta+\phi)/2)} = \frac{\sin(\phi/2)}{\sin((\theta+\phi)/2)} \quad \text{(by extracting and cancelling signs)}$$

I am reasoning that $\sin(\phi/2)$ and $\sin((\theta+\phi)/2)$ must both be positive or both negative. It is hard to completely reconcile this with [30b].

$$\frac{c-a}{b-a} = \left[\frac{\sin(\phi/2)}{\sin((\theta+\phi)/2)}\right] e^{-i\theta/2}$$

$$\text{c-a} = \left[\frac{\sin(\phi/2)}{\sin((\theta+\phi)/2)}\right] e^{-i\theta/2}\ \text{(b-a)}$$

$$= \left[\frac{b\sin(\phi/2)-a\sin(\phi/2)}{\sin(\theta+\phi)/2)}\right] e^{-i\theta/2}$$

$$\text{c} = \frac{(b\sin(\phi/2)-a\sin(\phi/2))\, e^{-i\theta/2}}{\sin(\theta+\phi)/2)} + \text{a}$$

Multiply numerator and denominator of first term by $(e^{i\phi/2}\, e^{i\theta/2})$. Multiply a by $\frac{1-e^{i(\theta+\phi)}}{1-e^{i(\theta+\phi)}}$.

$$= \frac{-2\,i\,(b\sin(\phi/2)-a\sin(\phi/2))\, e^{-i\theta/2}\left(e^{i\phi/2}\, e^{i\theta/2}\right)}{-2\,i\sin(\theta+\phi)/2)\left(e^{i\phi/2}\, e^{i\theta/2}\right)} + \frac{a\left(1-e^{i(\theta+\phi)}\right)}{1-e^{i(\theta+\phi)}}$$

$$= \frac{\left(-2\,i\,b\sin\frac{\phi}{2}\, e^{i\phi/2} - \left(-2\,i\,a\sin\frac{\phi}{2}\, e^{i\phi/2}\right)\right) + a\left(1-e^{i(\theta+\phi)}\right)}{-2\,i\sin(\theta+\phi)/2)\left(e^{i\phi/2}\, e^{i\theta/2}\right)}$$

$$= \frac{b\left(1-e^{i\phi}\right)- a\left(1-e^{i\phi}\right)+a\left(1-e^{i(\theta+\phi)}\right)}{1-e^{i(\theta+\phi)}} \qquad \text{[sine substitution]}$$

$$= \frac{b\left(1-e^{i\phi}\right)- a+ ae^{i\phi}+a-ae^{i(\theta+\phi)}\right)}{1-e^{i(\theta+\phi)}}$$

$$c = \frac{b\left(1-e^{i\phi}\right)+ae^{i\phi}\left(1-e^{i\theta}\right)}{1-e^{i(\theta+\phi)}} = \frac{ae^{i\phi}\left(1-e^{i\theta}\right)+b\left(1-e^{i\phi}\right)}{1-e^{i(\theta+\phi)}}$$

QED