Needham, Chapter 1, Exercise 34, p. 51
G. Palmer, April 19, 2020, 10:30 am PST

35 Show algebraically that the formula (21) is invariant under a translation by k, i.e., its value does not change if a + k, b becomes b + k, etc. Deduce from [22a] that the formula always gives the area of the quadrilateral. [Hint: Remember, (z+$\bar{z}$) is always real.]

Formula (21) is

$$\mathcal{A} = \tfrac{1}{2}\mathrm{Im}[\bar{a}\,\mathrm{b} + \bar{b}\,\mathrm{c} + \bar{c}\,\mathrm{d} + \bar{d}\,\mathrm{a}]$$

Under translation, every vector is translated.

$$\mathcal{A} = \tfrac{1}{2}\mathrm{Im}[(\bar{a}+\bar{k})\,(\mathrm{b}+\mathrm{k}) + \ldots]$$

Expand the first term:

$$(\bar{a}+\bar{k})(\mathrm{b}+\mathrm{k})] = \bar{a}\,\mathrm{b} + \mathrm{k}\,\bar{a} + \bar{k}\,\mathrm{b} + \;|k|^2$$

Proceed in this way through $\bar{b}\,\mathrm{c} + \bar{c}\,\mathrm{d} + \bar{d}\,\mathrm{a}$

$$(\bar{b}+\bar{k})(\mathrm{c}+\mathrm{k}) = \bar{b}\,\mathrm{c} + \mathrm{k}\,\bar{b} + \bar{k}\,\mathrm{c} + \;|k|^2$$

$$(\bar{c}+\bar{k})(\mathrm{d}+\mathrm{k}) = \bar{c}\,\mathrm{d} + \mathrm{k}\,\bar{c} + \bar{k}\,\mathrm{d} + \;|k|^2$$

$$(\bar{d}+\bar{k})(\mathrm{a}+\mathrm{k}) = \bar{d}\,\mathrm{a} + \mathrm{k}\,\bar{d} + \bar{k}\,\mathrm{a} + \;|k|^2$$

Add the terms and factor out k and $\bar{k}$.

$$\mathcal{A} = \tfrac{1}{2}\mathrm{Im}[\bar{a}\,\mathrm{b} + \bar{b}\,\mathrm{c} + \bar{c}\,\mathrm{d} + \bar{d}\,\mathrm{a} + \mathrm{k}\,(\overline{a + b + c + d}) + \bar{k}\,(\mathrm{a} + \mathrm{b} + \mathrm{c} + \mathrm{d}) + 4\,|k|^2]$$

Since k ($\overline{a + b + c + d}$) is the conjugate of $\bar{k}$ (a + b + c + d), their sum is real by the hint. The last term $4\,|k|^2$ is also real by the definition of the norm. Therefore,

$$\mathcal{A} = \tfrac{1}{2}\mathrm{Im}[\bar{a}\,\mathrm{b} + \bar{b}\,\mathrm{c} + \bar{c}\,\mathrm{d} + \bar{d}\,\mathrm{a}]$$

This shows that *formula (21) is invariant under a translation by k, i.e., its value does not change if a + k, b becomes b + k, etc.*

Deduce from [22a] that the formula always gives the area of the quadrilateral. [Hint: Remember, (z+$\bar{z}$) is always real.]

From the algebraic exercise we see that the area of the quadrilateral by this formula is invariant

under translation, so we can place the quadrilateral at any point in ℂ and the formula will apply. Does this deduction apply more aptly to [22b] than to [22a]?