Needham, Chapter 1, Exercise 33, p. 51
G. Palmer, March 3, 2020, 3:10 pm PST

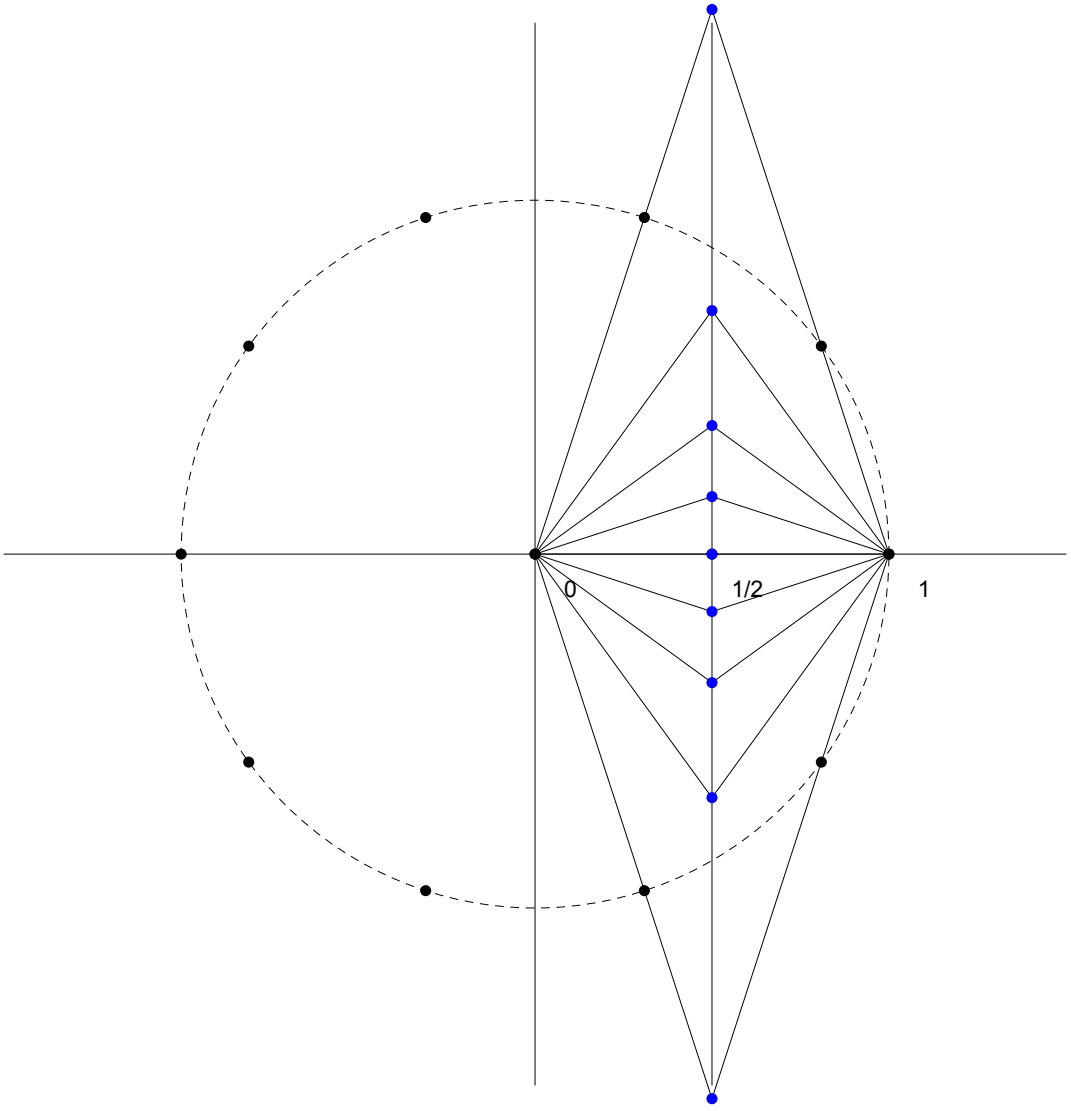


Figure 1. Solutions to $(z-1)^{10} = z^{10}$

33 Consider the equation $(z-1)^{10} = z^{10}$.

(i) Without attempting to solve the equation, show geometrically that all 9 solutions [why not 10?] must lie on the vertical line, $Re(z) = \frac{1}{2}$. *[Hint: Ex. 7].*

Why not 10? z = 1/2 is obviously a solution. Since it lies on the real line, we can guess that the other solutions are symmetrical about the x-axis. That is, the roots are divided into conjugate pairs. On the real line there is only a single solution: 1/2. Four above plus four below plus the solution on the real

line equals nine. But this is all guesswork.

Vasco has provided a algebraic explanation without solving the equation for its roots. He reasoned that $(z-1)^{10} = z^{10} + P_9(z)$. Then, since

$$z^{10} + P_9(z) = z^{10}$$

we know that $P_9(z) = 0$, so $P_9(z)$ has nine roots. Since 0 cannot be a solution, the nine roots of $P_9(z)$ are the only admissible solutions.

Then, I think, since we find empirically (see below) that the solutions all involve roots in conjugate pairs, except for the solution on the real line, it must be the case that $P_9(z) = (z - c_1)(z-\overline{c_1}), (z - c_2)(z-\overline{c_2}),..., (z - c_5)(z-\overline{c_5})$, and that c_3 is real, so that $c_3 = \overline{c_3}$, as speculated in the first paragraph, or that there is truly only one root on the real line, so that $P_9(z) = (z - c_1)(z-\overline{c_1}), (z - c_2)(z-\overline{c_2}), (z - c_3), (z - c_4)(z-\overline{c_4}), (z - c_5)(z-\overline{c_5})$. From Vasco's answer and the discovery that the 0 position on the regular n-gon cannot be used to find a solution, it seems that there is no conjugate pair for z on the real line, so my speculation in the first paragraph was unfounded. $P_9(z)$ has one real root and four pairs of conjugate roots. We also note that $P_9(z)$ cannot have the form $(z^9 = 1)$, for then the solution points would fall on the unit circle.

The equation $(z-1)^{10} = z^{10}$ entails that |z-1| = |z-0| = |z|. Since 1 and 0 are fixed numbers, |z-1| = |z| is the equation of a line, and the solutions must lie on this line. From Exercise 7, we can see that the line is orthogonal to the line between the fixed numbers 1 and 0 at the midpoint, which is 1/2.

(ii) Dividing both sides by z^{10}, the equation takes the form $w^{10} = 1$, where $w = (z-1)/z$. Hence, solve the original equation.

The solutions for w are the vertices of a regular decagon: $C_{k+1} = e^{ik(2\pi/n)}$, where n = 10 and k varies from 0 to 9.

$$w = \left\{1, e^{\frac{i\pi}{5}}, e^{\frac{2i\pi}{5}}, e^{\frac{3i\pi}{5}}, e^{\frac{4i\pi}{5}}, -1, e^{-\frac{4i\pi}{5}}, e^{-\frac{3i\pi}{5}}, e^{-\frac{2i\pi}{5}}, e^{-\frac{i\pi}{5}}\right\}$$

These points are plotted on the unit circle in Figure 1.

Now solve for z using z = –1/(w–1). Notice that we cannot use the first value w = 1 unless we wish to include ∞ as a 10th solution.

$$\left\{-\frac{1}{-1+e^{\frac{i\pi}{5}}}, -\frac{1}{-1+e^{\frac{2i\pi}{5}}}, -\frac{1}{-1+e^{\frac{3i\pi}{5}}}, -\frac{1}{-1+e^{\frac{4i\pi}{5}}}, \frac{1}{2}, -\frac{1}{-1+e^{-\frac{4i\pi}{5}}}, -\frac{1}{-1+e^{-\frac{3i\pi}{5}}}, -\frac{1}{-1+e^{-\frac{2i\pi}{5}}}, -\frac{1}{-1+e^{-\frac{i\pi}{5}}}\right\}$$

These points are plotted in blue on the vertical line Re(z) = 1/2 in Figure 1.

(iii) Express these solutions in the form z = x + iy, and thereby verify the result in (i). [Hint: To do this neatly, use Ex. 18.]

{ 0.5+1.53884 i, 0.5+0.688191 i, 0.5+0.363271 i, 0.5+0.16246 i, 0.5, 0.5-0.16246 i, 0.5-0.363271 i, 0.5-0.688191 i, 0.5-1.53884 i}

The result in (i) is verified by the fact that every point falls on the vertical line Re(z) = 0.5, but the problem calls for an algebraic solution. In an attempt to apply the equations in Exercise 18, we write the general equation for the solutions of z.

$$z = -\frac{1}{e^{in\pi/5}-1} = \frac{1}{1-e^{in\pi/5}} = \frac{1}{e^{i\cdot 0}-e^{in\pi/5}}$$

Then, using Exercise 18, let $\theta = 0$ and $\phi = n\pi/5$, and rewrite the denominator

$$e^{i\cdot 0} - e^{in\pi/5} = 2i \sin\left(\frac{0-n\pi}{5}\right) e^{i\left(\frac{0+n\pi}{5}\right)}$$

Then

$$z = -\frac{1}{2\, i \sin\left(\frac{n\pi}{5}\right) e^{i\left(\frac{n\pi}{5}\right)}} = -\frac{e^{-i\left(\frac{n\pi}{5}\right)}}{2\, i \sin\left(\frac{n\pi}{5}\right)} = \frac{\left(\cos\left(\frac{-n\pi}{5}\right)+i\sin\left(\frac{-n\pi}{5}\right)\right)}{2\, i \sin\left(-\frac{n\pi}{5}\right)} \qquad [\sin(-\theta) = -\sin(\theta)]$$

$$= \frac{-i\cos\left(\frac{-n\pi}{5}\right)}{2\sin\left(-\frac{n\pi}{5}\right)} + \frac{\sin\left(\frac{-n\pi}{5}\right)}{2\sin\left(-\frac{n\pi}{5}\right)} \quad \text{[multiply top and bottom by -i, separate real and imaginary]}$$

$$\text{Re}(z) = \frac{1}{2} \qquad \text{[cancel top and bottom of real term]}$$

This verifies that the nine solutions all lie on the vertical lines Re(z) = 1/2.