**Needham, Chapter 1, Exercise 32, p. 50-51.**
**G. Palmer, February 25, 2020, 8 am**

*32. The binomial theorem says that if n is a positive integer,* $(a+b)^n = \sum_{r=0}^{n} \binom{n}{r} a^{n-r} b^r$, *where* $\binom{n}{r} = \frac{n!}{(n-r)!\,r!}$ *are the binomial coefficients [not vectors!]. The algebraic reasoning leading to this result is equally valid if a and b are complex numbers. Use this fact to show that if n = 2m is even then*

$$\binom{2m}{1} - \binom{2m}{3} + \binom{2m}{5} - \ldots + (-1)^{m+1} \binom{2m}{2m-1} = 2^m \sin\left(\frac{mPi}{2}\right). \qquad [1]$$

After my first attempt and my first real answer, I read Vasco's answer and realized it was much more concise. I attempt to follow his approach without copying it directly.

Note that if we write

$$\mathrm{Im}[(a+b)^{2m}] = 2^m \sin\left(\frac{mPi}{2}\right) = \mathrm{Im}[2^m \, \mathrm{i} \sin(\tfrac{m\pi}{2}))] = \mathrm{Im}\left[\left(\sqrt{2}\, e^{\mathrm{i}\pi/4}\right)^{2m}\right]$$

then it must be the case that $(a+ib) = \sqrt{2}\, e^{\mathrm{i}\pi/4}$, showing that $|a+ib| = \sqrt{2}$, which is satisfied by $(\pm 1 \pm \mathrm{i})$, but only $(1+\mathrm{i})$ has the desired angle $\pi/4$. The calculations following this answer demonstrate that only $(1+\mathrm{i})$ satisfies the alternation of signs shown in the LHS of [1]. Note that if m is even, $\sin\left(\frac{\mathrm{m}\pi}{4}\right)$ vanishes. Using the binomial theorem,

$$(1+i)^{2m} = \sum_{r=0}^{2m} \binom{2m}{2r} 1^{2m-r}\, i^{2r}$$

$$= \sum_{r=0}^{m} \binom{2m}{2r} i^{2r} + \sum_{r=0}^{m-1} \binom{2m}{2r+1} i^{2r+1} \qquad \text{(omit powers of 1 and separate by even and odd r)}$$

$$= \mathrm{i} \sum_{r=0}^{m} \binom{2m}{2r} i^{2r-1} + i \sum_{r=0}^{m-1} \binom{2m}{2r+1} i^{2r} \qquad \text{(move one i outside the argument to Sigma)}$$

Setting the imaginary parts of the LHS and the RHS equal, and noting that the parts with 2r in the expansion must be real, we can write,

$$\sum_{r=0}^{m} \binom{2m}{2r+1} i^{2r} = \sum_{r=0}^{m} \binom{2m}{2r+1} (-1)^r = 2^m \sin\left(\frac{mPi}{2}\right) \qquad (2)$$

which can be rewritten as

$$\binom{2m}{1} - \binom{2m}{3} + \binom{2m}{5} - \ldots + (-1)^{m+1} \binom{2m}{2m-1} = 2^m \sin\left(\frac{mPi}{2}\right)$$

This is what we wanted.

We check equation (2) for m from 1 to 3.

m = 1

$$\frac{2!}{1!\ \ 0!}\cdot -1^0 = 2 = 2^1 \sin(\pi/2)$$

m = 2

$$\frac{4!}{(4-(2\cdot 0+1))!\ 1!}\cdot\left(-1^0\right)+\frac{4!}{(4-(2\cdot 1+1))!\ (2\cdot 1+1)!}\cdot(-1)^1 = 4-4 = 0 = 4\sin(\pi)$$

m = 3

$$\frac{6!}{(6-(2\cdot 0+1))!\ 1!}\cdot\left(-1^0\right)+\frac{6!}{(6-(2\cdot 1+1))!\ (2\cdot 1+1)!}\cdot\left(-1^1\right)+\frac{6!}{(6-(2\cdot 2+1))!\ (2\cdot 2+1)!}\cdot\left(-1^2\right) = 6-6+$$

$$= 6+\frac{6!}{3!\ \ 3!}\cdot\left(-1^1\right)+\frac{6!}{(6-5)!\ 5!}\cdot\left(-1^2\right) = 6-20+6 = -8 = 2^3\sin(3\pi/2)$$

### Calculations of $a^{2m-r}\,b^r$

a = 1, b = i, m = 9, r = 1,3,5,7,9:

$\{i, -i, i, -i, i\}$

a = i, b = 1, m = 9, r = 1,3,5,7,9:

$\{i, -i, i, -i, i\}$

a = b = i, m = 9, r = 1,3,5,7,9:

$\{-1, -1, -1, -1, -1\}$

a = b = 1, m = 9, r = 1,3,5,7,9:

$\{1, 1, 1, 1, 1\}$

a = i, b = −i, m = 9, r = 1,3,5,7,9:

$\{1, 1, 1, 1, 1\}$

a = −i, b = 1, m = 9, r = 1,3,5,7,9:

$\{-i, i, -i, i, -i\}$

a = −i, b = −i, m = 9, r = 1,3,5,7,9:

$\{-1, -1, -1, -1, -1\}$

a = 1.1, b = 1.1i, m = 9, r = 1,3,5,7,9:

$\{0. + 5.55992\, i, 0. - 5.55992\, i, 0. + 5.55992\, i, 0. - 5.55992\, i, 0. + 5.55992\, i\}$

a = .9+.9i, b = .8+.8i, m = 9, r = 1,3,5,7,9:

$\{8.53872 + 59.771\, i, 15.3908 + 33.944\, i, 14.9873 + 17.4549\, i, 11.8983 + 7.75321\, i, 8.39089 + 2.53801\, i\}$