Needham, Chapter 1, Exercise 31, p. 50
Palmer, February 19, 2020, 7:50 am PST

Table 1. $z^{n+2} = 2(z^{n+1} - z^n)$

2, 4, 4, 0
−8, −16, −16, 0
32, 64, 64, 0
−128, −256, −256, 0
512, 1024, 1024, 0
−2048, −4096, −4096, 0
8192, 16384, 16384, 0
−32768, −65536, −65536, 0

31 With the same recurrence relation as in the previous exercise, use a computer to generate the first 30 members of the sequence given by $S_1 = 2$ and $S_2 = 4$. Note the repeating pattern of zeros.

[It proved easier to generate and display the first 32.]

(i) With the same notation as before, show that this sequence corresponds to $A = -i$, so that $S_n = 2\,Re[-i(1 + i)^n]$.

Let A = a + ib.

$$S_n = 2\,\mathrm{Re}[(a + \mathrm{ib})\,(1 + i)^n];$$

If n = 1,

$$S_1 = 2 = 2\,\mathrm{Re}[(a + \mathrm{ib})\,(1 + i)] = 2\mathrm{Re}[\mathrm{a+ib+ai-b}] = 2\,\mathrm{Re}[(\mathrm{a-b}) + \mathrm{i}\,(\mathrm{a+b})]$$

$$= 2\,(\mathrm{a-b}), \quad \mathrm{a-b} = 1, \qquad \mathrm{b} = \mathrm{a-1}$$

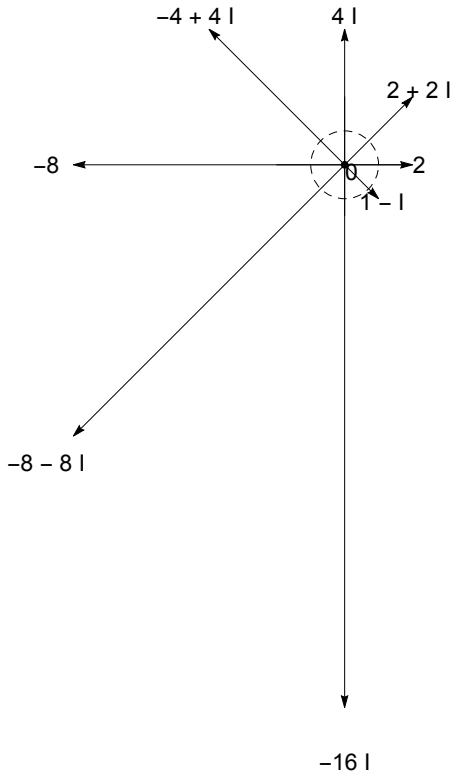
$$S_2 = 4 = 2\,\mathrm{Re}\left[(a + \mathrm{ib})\,(1 + i)^2\right] = 2\mathrm{Re}[(\mathrm{a+ib})\ 2\mathrm{i}] = 2\,(-2\mathrm{b}) = -4\mathrm{b}$$

$$\mathrm{b} = -1, \qquad \mathrm{a-1} = -1, \qquad \mathrm{a} = 0$$

$$\mathrm{A} = 0 - \mathrm{i} = -\mathrm{i}$$

Figure 1. Locations of $-i(1+i)^n$ for n = 1..8

Out[24]=



(ii) Draw a sketch showing the locations of $-i(1 + i)^n$ for n = 1 to n = 8, and hence explain the pattern of zeros.

The arrows in figure 1 show that the locations *of* $-i(1 + i)^n$ fall on the y–axis occur every fourth point. The angle of $(1 + i)$ is $\pi/4$, so every increment to n increments this angle by $\pi/4$. The result is that

$$\arg(-i(1 + i)^n) = ((n\pi/4) - \pi/2).$$

Starting at n = 0, every fourth point falls on the y–axis, where $\mathrm{Re}[-i(1 + i)^n]$ vanishes.

(iii) Writing A = a + ib, our example corresponds to a = 0. More generally, explain geometrically why such a repeating pattern of zeros will occur if and only if (a/b) = 0, ±1 or b = 0.

I think we must stipulate that a and b are real numbers. The first four steps of $-i(1 + i)^n$ are

$$1–i, 2, 2+2i, 4i$$

Every 2nd multiple of n, $-i(1 + i)^n$ falls on the real line (2nd step) or the y–axis (4th step). Then it takes another 4 multiples to return to the real line or to return to the y–axis.

(a/b) = 0 implies that a = 0, which means that $Re[b(-i(1 + i)^n)] = 0$, whenever $-i(1 + i)^n$ falls on the

y–axis, which is every time n is incremented by four.

When (a/b) = ±1, a = ±b. Whenever $-i(1+i)^n$ falls on the y–axis, it is expanded by ±b. Then if the sign of (±ib($-i(1+i)^n$) is negative, $Re[(a-ib)(1+i)^n] = 0$. Again, this happens every time n is incremented by four because the arrow reverses direction, showing that the sign of $-i(1+i)^n$ changes with every increment of n by four.

When b = 0, we need consider only Re[a(–i)$(1+i)^n$]. Whenever $-i(1+i)^n$ falls on the y–axis, the result is 0. Again, this happens every time n is incremented by four.

We have shown that a repeating pattern of zeros will occur if (a/b) = 0, ±1 or b = 0. Zeros can occur only if $-i(1+i)^n$ falls on the x or the y axis, which can occur, at most, every second step and it can only occur (1) if a = ±b so that the sum of terms inside the Re[] brackets equals 0, or (2) if a or b is zero. In every case with fixed A, this happens every fourth step. The spiral pattern in Figure 2 shows that opposite numbers (defined by Arg[z] = Arg[w] ± π) cannot generally sum to zero. Therefore we can also conclude that the repeating pattern of zeros every fourth number can occur **if and only if** (a/b) = 0, ±1 or b = 0.

(iv) Show that $\frac{S_1}{S_2} = \frac{1}{2}\left[1-\frac{a}{b}\right]$, and deduce that a repeating pattern of zeros will occur if and only if $S_2 = 2\,S_1$ (as in our example), $S_1 = S_2$, $S_1 = 0$, or $S_2 = 0$.

$$S_1 = 2\,\text{Re}[(a+ib)(1+i)] = 2\text{Re}[a+ia+ib-b] = 2\text{Re}[(a-b) + i(a+b)] = 2\,(a-b)$$

$$S_2 = 2\,\text{Re}\left[(a+\text{ib})\,(1+i)^2\right] = 2\,\text{Re}[(a+\text{ib})\,(2\,i)] = 2\,\text{Re}[-2\text{b} + 2\text{ai}] = -4\text{b}$$

$$\frac{S_1}{S_2} = \frac{2\,(a-b)}{-4\,b} = -\frac{(a-b)}{2\,b} = \frac{1}{2}\left(-\frac{a}{b} + 1\right) = \frac{1}{2}\left[1-\frac{a}{b}\right]$$

If $S_2 = 2\,S_1$, then

$$\frac{S_1}{S_2} = \frac{S_1}{2\,S_1} = \frac{1}{2}\left[1-\frac{a}{b}\right] \Longrightarrow \frac{1}{2} = \frac{1}{2}\left[1-\frac{a}{b}\right] \Longrightarrow \frac{a}{b} = 0 \Longrightarrow \mathbf{a = 0}$$

If $S_1 = S_2$, then

$$\frac{S_1}{S_2} = \frac{S_1}{S_1} = 1 = \frac{1}{2}\left[1-\frac{a}{b}\right] \Longrightarrow \left(1-\frac{a}{b}\right) = 2 \Longrightarrow -\frac{\text{a}}{\text{b}} = 1 \Longrightarrow \mathbf{\frac{a}{b} = -1}$$

If $S_1 = 0$, then

$$\frac{S_1}{S_2} = 0 \Longrightarrow \frac{1}{2}\left[1-\frac{a}{b}\right] = 0 \Longrightarrow \mathbf{\frac{a}{b} = 1}$$

Here we see also that if b = 0, then a = 0 · 1 = 0.

If $S_2 = 0$, then

$\frac{S_1}{S_2} = \frac{1}{2}\left[1 - \frac{a}{b}\right] \Longrightarrow S_1 = S_2\ \frac{1}{2}\left[1 - \frac{a}{b}\right] = 0$, which from the previous, implies $\boldsymbol{\frac{a}{b} = 1}$.

In part (iii), we showed that the repeating pattern of zeros every fourth number in the series can occur only if and only if (a/b) = 0, ±1 or b = 0. Given the constraints, $S_2 = 2\,S_1$ (as in our example), $S_1 = S_2$, $S_1 = 0$, or $S_2 = 0$, we have satisfied the constraint that $\frac{a}{b} = \pm 1$, which was one of the constraints for a repeating pattern of zeros. Suppose that none of the constraints $S_2 = 2\,S_1$ (as in our example), $S_1 = S_2$, $S_1 = 0$, or $S_2 = 0$ holds.

Let $S_1 = nS_2$, where n ≠ 1 or 2.

$$\frac{S_1}{S_2} = \frac{S_1}{k\,S_1} = \frac{1}{2}\left[1 - \frac{a}{b}\right] \Longrightarrow\ \frac{1}{2}\left[1 - \frac{a}{b}\right] = \frac{1}{k}, \left[1 - \frac{a}{b}\right] = \frac{2}{k}, -\frac{a}{b} = \frac{2-k}{k}, \boldsymbol{\frac{a}{b} = \frac{k-2}{k}}$$

This does not match any of the criteria established in part (iii). It matches $\frac{a}{b} = \pm 1$ and $\frac{a}{b} = 0$ only if k = 1 or 2, which contradicts our premise.

Suppose $S_1 \neq 0$, then $\frac{1}{2}\left[1 - \frac{a}{b}\right] = 0$ is true only where $\frac{a}{b} = 1$, but that happens only where we let S_1 or S_2 equal zero. We conclude that a repeating pattern of zeros will occur if and only if $S_2 = 2\,S_1$ (as in our example), $S_1 = S_2$, $S_1 = 0$, or $S_2 = 0$.

(v) Use a computer to verify these predictions.

Criteria for pattern of vanishing zeros: (a/b)=0, ±1 or b = 0

OR $S_1 = S_2,\ S_1 = 0,\ \text{or}\ S_2 = 0$

a=0, b=0.5, a/b=0.

series 2 Re[(a+ib)(1+I)^n: {−1., −2., −2., 0., 4., 8., 8., 0., −16., −32., −32., 0., 64., 128., 128., 0., −256., −512., −512., 0., 1024., 2048., 2048., 0., −4096., −8192., −8192., 0., 16 384., 32 768.}

S_1=−1., S_2=−2., S_2=2S_1

a=0.5, b=0

series 2 Re[(a+ib)(1+I)^n: {1., 0., −2., −4., −4., 0., 8., 16., 16., 0., −32., −64., −64., 0., 128., 256., 256., 0., −512., −1024., −1024., 0., 2048., 4096., 4096., 0., −8192., −16 384., −16 384., 0.}

S_2=0.

a=0.5, b=0.5, (a/b)=1

series 2 Re[(a+ib)(1+I)^n: {0., −2., −4., −4., 0., 8., 16., 16., 0., −32., −64., −64., 0., 128., 256., 256., 0., −512., −1024., −1024., 0., 2048., 4096., 4096., 0., −8192., −16 384., −16 384., 0., 32 768.}

S_1=0.

a: −0.5 b: 0.5 (a/b)=−1

series 2 Re[(a+ib)(1+I)^n: {−2., −2., 0., 4., 8., 8., 0., −16., −32., −32., 0., 64., 128., 128., 0., −256., −512., −512., 0., 1024., 2048., 2048., 0., −4096., −8192., −8192., 0., 16 384., 32 768., 32 768.}

S_1=−2., S_2=−2., S_1=S_2

When criteria are violated, pattern of repeating zeros vanishes.

a=−0.5, b=0.6, (a/b)=−0.833333

series 2 Re[(a+ib)(1+I)^n: {−2.4, −0.4, 4., 8.8, 9.6, 1.6, −16., −35.2, −38.4, −6.4, 64., 140.8, 153.6, 25.6, −256., −563.2, −614.4, −102.4, 1024., 2252.8, 2457.6, 409.6, −4096., −9011.2, −9830.4, −1638.4, 16 384., 36 044.8, 39 321.6}

S_1=−2.2, S_2=−2.4