Needham, Chapter 1, Exercise 30, p. 50
Palmer, February 14, 2020, 9:30 am PST

30 Given two starting numbers S_1, S_2, let us build up an infinite sequence $S_1, S_2, S_3, S_4, \ldots$ with this rule: each new number is twice the difference of the previous two. For example, if $s_1 = 1$ and $s_2 = 4$, we obtain 1, 4, 6, 4, -16, -24, ... Our aim is to find a formula for the n^{th} number S_n.

(i) Our generating rule can be written succinctly as $S_{n+2} = 2\,(S_{n+1} - S_n)$. Show that $S_n = z^n$ will solve this recurrence relation if $z^2 - 2z + 2 = 0$.

Rewrite the recurrence relation in terms of z^n.

$$z^{n+2} = 2\left(z^{n+1} - z^n\right)$$

$$= 2\,(\,z^n\,z - z^n) = 2\,z^n\,(z\text{-}1)$$

Let n = 0.

$$z^2 = 2\,(z\text{-}1) \qquad (1)$$

Subtract 2(z-1) from both sides and rearrange.

$$z^2 - 2z + 2 = 0$$

(ii) Use the quadratic formula to obtain $z = 1 \pm i$, and show that if A and B are arbitrary complex numbers, $S_n = A\,(1 + i)^n + B(1 - i)^n$ is a solution of the recurrence relation.

$$z^2 - 2z + 2 = 0$$

$$z = \frac{2 \pm \sqrt{4 - 4 \cdot 2}}{2} = \frac{2 \pm 2\sqrt{-1}}{2} = 1 \pm i$$

I will try a proof by induction. Now change the notation values a bit and let z = (1+i) so $\bar{z} = (1\text{-}i)$, and let $Z^n = Az^n + B\,\bar{z}^n$. Then we are trying to show that $Z^n = Az^n + B\,\bar{z}^n$ is a solution to the recurrence relations. Show the relation for Z^{n+2} where n = 0, the base case.

$$Z^{n+2} = (Az^2 + B\,\bar{z}^2) = A(1+2i\text{-}1) + B(1\text{-}2i\text{-}1)$$

$$= A + 2iA - A + B - 2iB - B = 2i(A\text{-}B) \qquad (2)$$

$$2\left(Z^{n+1} - Z^n\right) = 2[A(1 + i) + B(1 - i)\text{ -A-B}]$$

$$= 2[A+Ai+B\text{-}Bi\text{-}A\text{-}B] = 2i(A\text{-}B) \qquad (3)$$

Since (3) = (2), this shows that S_0 is a solution to the recurrence relation.

For the general case, let n = p, where p ∋ 0...∞.

$$Z^{p+2} = 2\left(Z^{p+1} - Z^{p}\right)$$

$$Z^{p+2} = \mathrm{Az}^{p+2} + B\,\overline{z}^{p+2}$$

$$2\left(Z^{p+1} - Z^{p}\right) = 2\,Z^{p}(\mathrm{Z\text{-}1})$$

$$= 2\,(\mathrm{Az}^{p} + B\,\overline{z}^{p})[(\mathrm{Az} + B\,\overline{z}) - 1]$$

$$= 2(\left[\mathrm{Az}^{p+1} + B\,\overline{z}^{p+1}\right] - [\mathrm{Az}^{p} + B\,\overline{z}^{p}])$$

$$= 2[\mathrm{Az}^{p+1} - \mathrm{Az}^{p}] + 2[B\,\overline{z}^{p+1} - B\,\overline{z}^{p}]$$

Let $P = \mathrm{Az}^{p}$ and let $Q = B\,\overline{z}^{p}$. One can see that P and Q are both solutions of the recurrence relation. Therefore, P + Q is also a solution, so $S_p = A\,(1 + i)^p + B(1 - i)^p$ is a solution.

Write the equation for p+1.

$$Z^{(p+1)+2} = 2[\mathrm{Az}^{(p+1)+1} - \mathrm{Az}^{(p+1)}] + 2[B\,\overline{z}^{(p+1)+1} - B\,\overline{z}^{(p+1)}]$$

$$Z^{p+3} = 2[\mathrm{Az}^{(p+2)} - \mathrm{Az}^{(p+1)}] + 2[B\,\overline{z}^{(p+2)} - B\,\overline{z}^{(p+1)}]$$

If S_p is a solution, S_{p+1} must also be a solution, and we know that S_p is a solution, and S_0 is a solution, so the recurrence relation is true for all $S_n = A\,(1 + i)^n + B(1 - i)^n$.

(iii) If we want only real solutions of the recurrence relation, show that $B = \overline{A}$*, and deduce that* $S_n = 2\,Re[A\,(1 + i)^n]$.

Show that

$$\mathrm{Re}[\mathrm{A}\,(1 + i)^n] = \mathrm{Re}[\,\overline{A}\,(1 - i)^n]$$

Letting z = (1+i), $\overline{z}$ = (1-i), rewrite this as

$$\mathrm{Re}(\mathrm{A}\,z^n) = \mathrm{Re}(\,\overline{A}\,\overline{z}^n) = \mathrm{Re}(\overline{A\,z^n})$$

This is true, because Re(z) = Re($\overline{z}$) and $\overline{z}^n = \overline{z^n}$ for all complex numbers, so Re[A $(1 + i)^n$] + Re[B $(1 - i)^n$] = 2 Re[A $(1 + i)^n$] if B = $\overline{A}$. Therefore, $S_n = 2\,\mathrm{Re}[A\,(1 + i)^n]$ if only real solutions are desired.

(iv) Show that for the above example [in the problem introduction] $A = -(1/2) - i$*, and by writing this in polar form deduce that* $S_n = 2^{n/2}\,\sqrt{5}\,\cos\left[\frac{(n+4)\,\pi}{4} + \tan^{-1} 2\right]$.

Let A = a + bi

$S_1 = 1 = 2\ Re[A\ (1 + i)] \Longrightarrow$ Re[(a + b i) (1+i)] = Re[(a-b + i(a+b))] = 1/2 $\Longrightarrow$ (a-b) = 1/2

$S_2 = 4 = 2$ Re[A $(1 + i)^2] \Longrightarrow$ Re[(a+bi) (2i)] = Re[-2b+2ia] = 2, ==> -2b = 2, **b = -1, ==> a = -1/2**

Therefore, A = a + bi = -(1/2) - i.

In polar form, A = $re^{i\theta}$, where

$$r = \sqrt{(1/4) + 1} = \sqrt{5/4} = \sqrt{5}/2$$

and $\theta = \tan^{-1}(-1/-(1/2)) = \tan^{-1} 2 + \pi$, because it lies in the third quadrant and $\tan^{-1}$ is a multi-function with values between $-\pi/2 \le E \le \pi/2$. $\text{Tan}^{-1}(2) = 1.10715$, which is in the first quadrant, so we add π.

$$A = \sqrt{5}/2\ e^{i\left(\tan^{-1} 2+\pi\right)} \qquad (4)$$

Deduce that $S_n = 2^{n/2}\sqrt{5}\ cos\left[\frac{(n+4)\pi}{4} + tan^{-1} 2\right]$.

Rewrite $(1 + i)^n$ in polar format.

$$\left(\sqrt{2}\right)^n e^{i\,n \tan^{-1} 1} = \sqrt{2}^{\,n}\, e^{i\,n\,\pi/4} = 2^{n/2} e^{i\,n\,\pi/4} \qquad (5)$$

Substitute (4) and (5) into $S_n = 2$ Re[$A\ (1 + i)^n$].

$$S_n = 2\ \text{Re}[A\ (1 + i)^n] = 2\ \text{Re}[\left(\sqrt{5}\ /\ 2\right) e^{i\left(\tan^{-1} 2+\pi\right)}\, 2^{n/2}\, e^{i\,n\,\pi/4}]$$

$$= 2\ \text{Re}[2^{n/2}\left(\sqrt{5}\ /\ 2\right) e^{i\left(\tan^{-1} 2+ (n+4)\,\pi/4\right)}]$$

$$= 2\text{Re}[2^{n/2}\left(\sqrt{5}\ /\ 2\right)(\cos(\tan^{-1} 2 + (n+4)\,\pi/4) + \text{i}\,\sin(\tan^{-1} 2 + (n+4)\,\pi/4))]$$

$$= 2^{n/2}\sqrt{5}\ \cos((n+4)\,\pi/4 + \tan^{-1} 2) \qquad (6)$$

(v) Check that this formula predicts $S_{34} = 262144$*, and use a computer to verify this.*

The original formula 2 Re[A $(1 + i)^n$] correctly predicts the value with A = -(1/2) - i and n = 34. Equation (6) also predicts the number correctly. If we print S_n from n = 1..34, we see that each successive group of four numbers changes sign from that of the prior group, beginning with a positive sign for the first group, negative sign for the second group, etc..

$S_n = 2\mathrm{Re}[A(1+i)^n$

{1, 4, 6, 4}

{−4, −16, −24, −16}

{16, 64, 96, 64}

{−64, −256, −384, −256}

{256, 1024, 1536, 1024}

{−1024, −4096, −6144, −4096}

{4096, 16 384, 24 576, 16 384}

{−16 384, −65 536, −98 304, −65 536}

{65 536, 262 144, 393 216, 262 144}

{−262 144, −1 048 576, −1 572 864, −1 048 576}

$2^{n/2}\sqrt{5}\,\mathrm{Cos}[(n+4)\pi/4 + \tan^{-1}2$

{1., 4., 6., 4.}

{−4., −16., −24., −16.}

{16., 64., 96., 64.}

{−64., −256., −384., −256.}

{256., 1024., 1536., 1024.}

{−1024., −4096., −6144., −4096.}

{4096., 16 384., 24 576., 16 384.}

{−16 384., −65 536., −98 304., −65 536.}

{65 536., 262 144., 393 216., 262 144.}

$\{-262\,144.,\ -1.04858\times10^6,\ -1.57286\times10^6,\ -1.04858\times10^6\}$