Needham, Chapter 1, Exercise 3, p. 45.
G. Palmer, December 22, 2019, 1:50 pm PST

3 In 1591, more than forty years after the appearance of (4), Francois Viète published another method of solving cubics. The method is based on the identity (see p. 15) cos 3θ = 4 C^3-3C, where C = cosθ.

(i) Substitute $x = 2\sqrt{p}\,C$ into the (reduced) general cubic $x^3 = 3px + 2q$ to obtain $4C^3 - 3C = \frac{q}{p\sqrt{p}}$.

$$(2\sqrt{p}\,C)^3 = 3p\,(2\sqrt{p}\,C) + 2q$$

$$4\,p\sqrt{p}\,C^3 = 3p\sqrt{p}\,C + q \quad \text{(divide through by 2)}$$

$$(4C^3 - 3C)\,p\sqrt{p} = q \quad \text{(rearrange)}$$

$$(4C^3 - 3C) = \frac{q}{p\sqrt{p}} = \cos 3\theta \qquad (1)$$

(ii) Provided that $q^2 \le p^3$, deduce that the solutions of the original equation are

$$x = 2\sqrt{p}\,\cos\left[\tfrac{1}{3}(\phi + 2m\pi)\right],$$

where m is an integer and $\phi = \cos^{-1}(q / p\sqrt{p})$.

Since -1 ≤ cos ≤ 1, shouldn't the author have written "provided that $q^2 \le |p^3|$, so that $\left|\frac{q}{p\sqrt{p}}\right| \le 1$?

Then since $x = 2\sqrt{p}\,C = \cos\theta$, we substitute for ϕ.

$$C = \cos\left[\tfrac{1}{3}\left(\cos^{-1}(q / p\sqrt{p}) + 2m\pi\right)\right] \quad \text{(from (1))}$$

$$C = \cos\left[\tfrac{1}{3}(3\theta + 2m\pi)\right] = \cos\theta \quad \text{(taking only primary branch of cosine)}$$

Then we can also recover (1)

$$8p\sqrt{p}\cos^3\theta = 3\cdot 2\sqrt{p}\cos\theta + 2q$$

$$4p\sqrt{p}\cos^3\theta - 3p\sqrt{p}\cos\theta = q$$

$$4C^3 - 3C = \frac{q}{p\sqrt{p}}$$

which is a form of the reduced general cubic where $x = 2\sqrt{p}\,C$.

(iii) Check that this formula gives the correct solutions of $x^3 = 3x$, namely $x = 0, \pm 3$.

Using the parts of the general equation, $p = 1$ and $q = 0$.

$$x^3 = 3 \cdot 2\sqrt{p}\;C$$

$$= 6\cos\left[\frac{1}{3}(\phi + 2m\pi)\right]$$

$$= 6\cos\left[\frac{1}{3}\left(\cos^{-1}\left(q/p\sqrt{p}\right) + 2m\pi\right)\right]$$

$$= 6\cos\left[\frac{1}{3}\left(\cos^{-1}(0) + 2m\pi\right)\right]$$

$$= 6\cos\left[\frac{1}{3}\left(\pm\frac{\pi}{2} + 2m\pi\right)\right]$$

$$= 6\cos\left[\frac{\pi}{3}\left(\pm\frac{1}{2} + 2m\right)\right]$$

This gives values of $x^3 = -3\sqrt{3}, 0, 3\sqrt{3}$, which have the roots $x = -\sqrt{3}, 0, \sqrt{3}$.

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Table[CubeRoot[6 Cos[(Pi / 3) ((1 / 2) + 2 m)]], {m, -3, 3}]
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$$\left\{\sqrt{3}, -\sqrt{3}, 0, \sqrt{3}, -\sqrt{3}, 0, \sqrt{3}\right\}$$