Needham, Chapter 1, Exercise 29, p. 49.
Palmer, February 11, 2020, 8:40 am PST

In[294]:=
```
Clear["Global`*"]

(* GLOBALS *)

ptSz = PointSize[.007];

hdSz = Arrowheads[.015];

a = .6;

b = 1.5;

θMax = 5.5 Pi / 4;

(* FUNCTIONS *)

SetAttributes[{v}, Listable];

v[z_] := {Re @ z, Im @ z};

corner[z_, θ_, sc_] :=
    Rotate[{Line[v @ {z, z + sc I}], Line[v @ {z + sc I, z + sc I + sc 1}]}, θ, v @ z];

z[t_] := (1 / (a + I b)) Exp[a t] Exp[I b t];


(* OBJECTS *)

uC = {Dashed, Circle[v @ 0, 1]};

yPts = {I, -1.5 I};

xPts = {-2, 2};

yLine = {Thin, Line[v @ yPts]};

xLine = {Thin, Line[v @ xPts]};

spiralPts = Table[z[t], {t, -2 π, θMax / b, Pi / 50}];

spiral = {ptSz, Arrow[v @ spiralPts]};
```

```
vel = (a + I b) spiralPts[[-1]];

velRay = {hdSz, Arrow[v @ {0, vel}]};

tanRay = {hdSz, Arrow[v @ {spiralPts[[-1]], spiralPts[[-1]] + vel}]};

rayZ = Arrow[v @ {Exp[I Mod[θMax, (2 Pi)]], z}];

(* vector sum *)

sum = (1 / (a + I b)) (Exp[a] Exp[I b] - 1);

sumArr = {hdSz, Arrow[v @ {0, sum}]};

dt = .25;

tanPts = Table[Exp[a t] Exp[I b t] dt, {t, 0, 1, dt}];

tanTuples = Table[{tanPts[[n]], tanPts[[n + 1]]}, {n, 1, Length[tanPts] - 1}];

tanVecs = {hdSz, Arrow[v @ tanTuples]};

(* COLLECTIONS *)

lines = {yLine, xLine, velRay, tanRay, sumArr, tanVecs};

curves = {uC, spiral};

points = {ptSz, Point[v @ { 0, 1, sum}]};

(* TEXT *)

def = 12;

data = {
    {"0", def, + .2 - .2 I},
    {"V", def, spiralPts[[-1]] + vel / 2 - .2 - .2 I},
    {"V", def, vel / 2 - .2 - .2 I},
    {"(1/(a+ib)) (e^a e^ib-1)", def, sum + 1 + .1 I},
    {"Figure 1. Z(t) = (1/(a+ib))e^at e^ibt\na = " <>
        ToString[a] <> ", b = " <> ToString[b], def + 4, -4 I}
  };
```

```
text = Table[Text[Style[rec[[1]], rec[[2]]], v @ rec[[3]]], {rec, data}];

(* GRAPHICS *)

rng = 2;

plotrng = {{-6, 3}, {-6, 3}};

Graphics[{lines, curves, points, text},
  Background → White, PlotRange → plotrng, ImageSize → Large]
```

Out[329]=

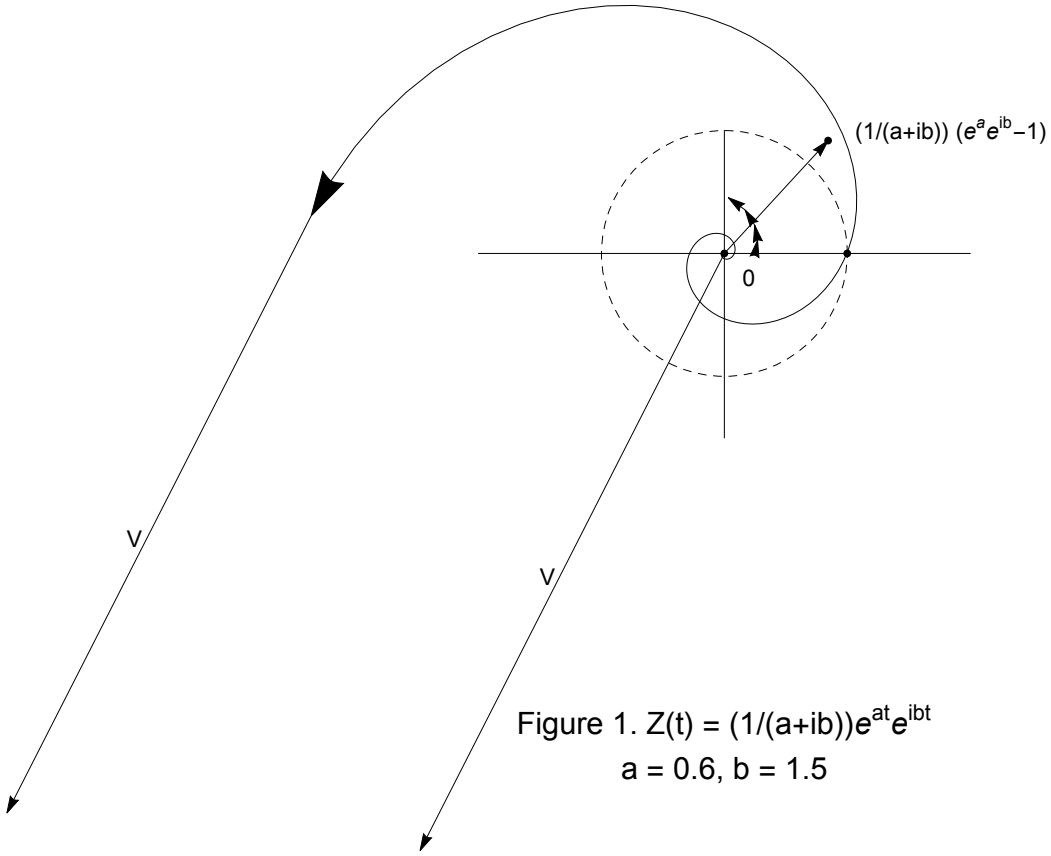


Figure 1. $Z(t) = (1/(a+ib))e^{at}e^{ibt}$
a = 0.6, b = 1.5

29 (i) If V(t) is the complex velocity of a particle whose orbit is Z(t), and dt is an infinitesimal moment of time, then V(t) dt is a complex number along the orbit. Thinking of the integral as the (vector) sum of these movements, what is the geometric interpretation of $\int_{t_1}^{t_2} V(t)\,dt$.

The vector sum can be interpreted in two ways: As a series of vectors V(t)dt arranged in order end to end, or (2) as the complex number or ray pointing to the head of the vector at the end of the series. My plots of these two do not match up, as explained in (iv).

(ii) Referring to [15b] sketch the curve Z(t) = $\frac{1}{a+ib}\left(e^{at}\, e^{ibt}\right)$. See Figure 1.

(iii) Given the result (13), what is the velocity of the particle in the previous part.

$$V(t) = (a+ib)Z(t) = (a+ib)\left(\frac{1}{a+ib} e^{at}\, e^{ibt}\right) = e^{at}\, e^{ibt}$$

(iv) Combine the previous parts to deduce that $\int_0^1 e^{at}\, e^{ibt}\, dt = \left[\frac{1}{a+ib}\, e^{at}\, e^{ibt}\right]_0^1$*, and draw in this complex number in your sketch for part (ii).*

Since $V(t) = (a+ib)Z(t) = e^{at}\, e^{ibt}$, it follows that

$$\int_0^1 V(t)\, dt = \int_0^1 e^{at}\, e^{ibt}\, dt = \left[\frac{1}{a+ib}\, V(t)\right]_0^1 = \left[\frac{1}{a+ib}\, e^{at}\, e^{ibt}\right]_0^1.$$

$$\left[\frac{1}{a+ib}\, e^{at}\, e^{ibt}\right]_0^1 = \left(\frac{1}{a+ib}\, e^{a}\, e^{ib} - \frac{1}{a+ib}\right) = \frac{1}{a+ib}\,(e^{a}\, e^{ib} - 1) \qquad (1)$$

PROBLEM: I have drawn in the complex number (1), and I have drawn in five "infinitesimal" velocity vectors for t = 0..1. I would have expected the end point of the velocity vectors to match (1).

(v) Use this to deduce that

$$\int_0^1 e^{at} \cos bt\, dt = \frac{a(e^{a}\cos b - 1) + b\, e^{a} \sin b}{a^2+b^2}$$

and

$$\int_0^1 e^{at} \sin bt\, dt = \frac{b(1 - e^{a}\cos b) + a\, e^{a} \sin b}{a^2+b^2}$$

$$\int_0^1 e^{at}\, e^{ibt}\, dt = \frac{1}{a+ib}\left(e^{a}\, e^{ib} - 1\right) = \int_0^1 e^{at}(\cos bt + i \sin bt)\, dt \qquad \text{[from (1) and expansion]}$$

$$= \int_0^1 e^{at} \cos bt\, dt + i \int_0^1 e^{at} \sin bt\, dt \qquad (2)$$

$$\frac{1}{a+ib}\,(e^{a}\, e^{ib} - 1) = \frac{1}{a+ib}\,(e^{a}\,(\cos b + i \sin b) - 1) = \frac{a-ib}{a-ib}\,\frac{1}{a+ib}\,(e^{a}\,(\cos b + i \sin b) - 1)$$

$$= \frac{a-\text{ib}}{a^2+b^2}(e^a\ (\cos \text{b} + \text{i} \sin \text{b}) - 1)$$

$$= \frac{a}{a^2+b^2}(e^a\ (\cos \text{b} + \text{i} \sin \text{b}) - 1) - \frac{\text{ib}}{a^2+b^2}((e^a \cos \text{b} + \text{i} \sin \text{b}) - 1)$$

$$= \frac{a(e^a \cos b - 1) + a\, e^a\, i \sin b}{a^2+b^2} + \frac{\text{ib}\,(1 - e^a \cos b) - \text{ib}\, e^a\, i \sin b}{a^2+b^2}$$

$$= \frac{a(e^a \cos b - 1) + \text{be}^a \sin b}{a^2+b^2} + \frac{\text{ib}\,(1 - e^a \cos b) + a\, e^a\, i \sin b}{a^2+b^2}$$

$$= \frac{a(e^a \cos b - 1) + \text{be}^a \sin b}{a^2+b^2} + \text{i}\ \frac{b\,(1 - e^a \cos b) + a\, e^a \sin b}{a^2+b^2} \qquad (3)$$

Now set the real and imaginary parts of (2) and (3) equal to obtain the desired equations.