**Needham, Chapter 1, Exercise 28, p. 47.**
**Palmer, February 11, 2020, 8 am PST**

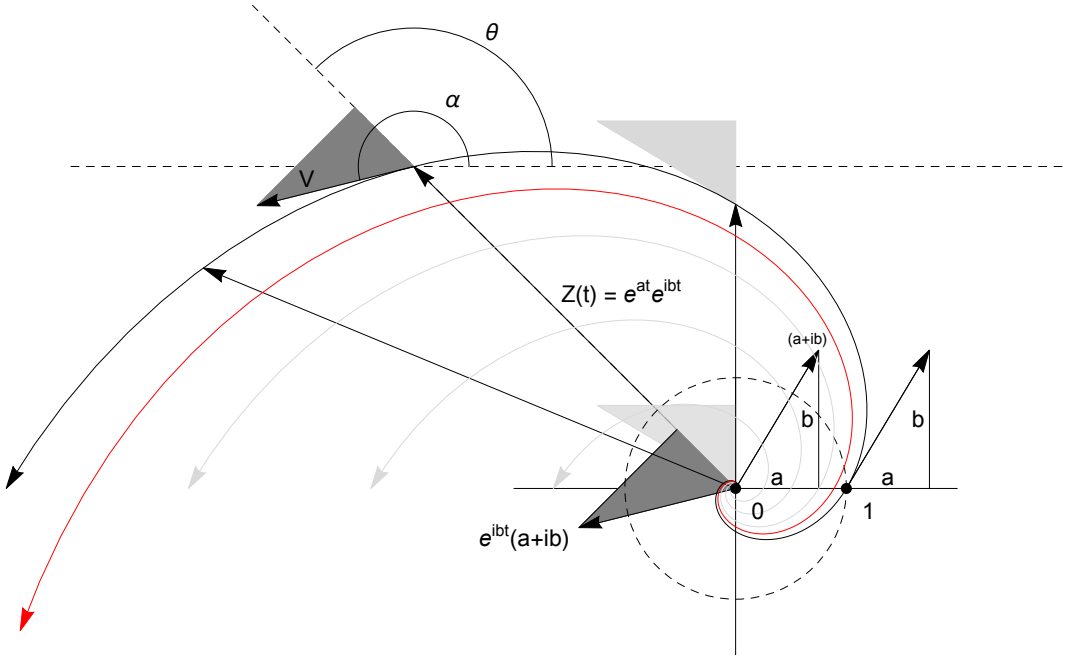



Figure 1. Spiral as invariant curve

*28 Reconsider the spiral $Z(t) = e^{at}\, e^{ibt}$ in [15b] [p. 20], where a and b are fixed real numbers. Let $\tau$ be a variable real number. According to (9), $z \mapsto \mathcal{F}_\tau = (e^{a\tau}\, e^{ib\tau})\, z$ is an expansion of the plane by factor $e^{a\tau}$, combined with a rotation of the plane through the angle bt.*

*(i) Show that $\mathcal{F}_\tau[Z(t)] = Z(t + \tau)$, and deduce that the spiral is an invariant curve (cf. p. 38) of the transformations $\mathcal{F}_\tau$.*

$$\mathcal{F}_\tau[Z(t)] = e^{a\tau}\, e^{ib\tau}\, Z(t) = e^{a\tau}\, e^{ib\tau}\, e^{at}\, e^{ibt}$$

$$= (e^{a\tau}\, e^{at})\left(e^{ib\tau} e^{ibt}\right) = e^{a(t+\tau)} e^{ib(t+\tau)} = Z(t + \tau) \qquad (1)$$

The exponents $a(t + \tau)$ and $ib(t + \tau)$ of $Z(t + \tau) = e^{a(t+\tau)} e^{ib(t+\tau)}$ both have fixed coefficients (a and b) and linear variables (t and $t + \tau$). If allowed to increase indefinitely, t and $(t+\tau)$ pass through all the same values. Therefore, the shape of the curve is entirely determined by a and b, and the spiral maps $Z(t)$ into $Z(t + \tau)$. Therefore, $Z(t)$ is an invariant curve of the transformation $\mathcal{F}_\tau$. If the spiral did not map into itself, one might see new spirals such as the light gray spirals in Figure 1 which have been scaled to have smaller radii at the same value of t.

*(ii) Use this to give a calculus-free demonstration that all rays from the origin cut the spiral at the same angle.*

Vasco makes the basic point that Z(t+$\tau$) involves a rotation of the plane plus an expansion of the plane compared to Z(t). Neither the rotation nor the expansion change the angle between the curve and the ray. This seems all one really needs to say, but I will add two more demonstrations. The first is based on geometry, the second on algebra and geometry.

In [16] and Figure 1, the triangle associated with $(a + ib)$ has a base which is a segment of the real line. When the triangle is rotated by bt with the ray Z(t), the base becomes a segment of Z(t). The vector (a + ib) has the same angle as the tangent to the spiral. The rotated diagonal can be translated from the origin to Z(t) where it becomes tangent to the curve. The triangle is subject to the rotation $\mathcal{R} = e^{ibt}$ plus the translation $\mathcal{T} = e^{ibt}(a + ib) \longrightarrow e^{ibt}(a + ib) + Z(t)$. Since the triangle and Z(t) rotate together by the same angle, the tangent (hypoteneuse) remains at the same angle to the ray Z(t), showing that all rays from the origin cut the spiral at the same angle. The reader is cautioned not to think of the tangent $e^{ibt}(a + ib)$ as a derivative, because that would be calculus and this demonstration must be calculus-free. This is just finding the tangent from the slope of (a+ib). The tangent found in this way does share the angle of the derivative at time t, but it does not necessarily have the same length.

The second argument is from algebra and geometry:

Let $\theta$ = bt and let $\phi$ = b$\tau$. Find the angle between the ray $Z(t) = e^{at}e^{ibt} = e^{at}e^{i\theta}$ and its tangent. Then make a new ray $Z(t+\tau) = e^{a(t+\tau)}e^{ib(t+\tau)} = e^{a(t+\tau)}e^{i(\theta+\phi)}$ and find the angle with its tangent. The tangent of Z(t) is $(a+ib)e^{i\theta}$. The tangent of Z(t+$\tau$) is $(a+ib)\, e^{i(\theta+\phi)}$. Then we are trying to prove that the angle on the left below is equal to the angle on the right.

$$\arg[(a+ib)e^{i(\theta+\phi)}] - (\theta+\phi) = \arg[(a+ib)e^{i\theta}] - \theta$$

$$\arg(a+ib)+\arg(e^{i(\theta+\phi)}) - (\theta+\phi) = \arg(a+ib) + \arg(e^{i\theta}) - \theta$$

$$\arg(a+ib)+(\theta+\phi) - (\theta+\phi) = \arg(a+ib) + \theta - \theta$$

$$\arg(a+ib) = \arg(a+ib)$$

showing that the two angles are equal.

*(iii) Show that if the spiral is rotated about the origin through an arbitrary angle, the new spiral is again an invariant curve of each $\mathcal{F}_\tau$.*

The red spiral in Figure 1 rotates the original black spiral by $\pi/16$. I read the question as: Show that if the spiral Z(t) is rotated about the origin through an arbitrary angle, the new spiral is again an invariant curve under $\mathcal{F}_\tau$. By that, I mean that $\mathcal{F}_\tau\left[Z(t)\, e^{i\theta}\right]$ maps $Z(t)\, e^{i\theta}$ into itself.

Let $\mathcal{W}(t) = Z(t)\, e^{i\theta}$, where $\theta$ is an arbitrary angle. We can write, by substitution for Z(t) from (1):

$$\mathcal{F}_\tau[W(t)] = \mathcal{F}_\tau\left[Z(t)\, e^{i\theta}\right] = e^{a(\tau+t)}e^{ib(\tau+t)}e^{i\theta} = Z(\tau+t)\, e^{i\theta} = W(\tau+t)$$

Since $\theta$ is constant, the new rotated spiral is still governed in shape by the constants a and b, which are scaled by the variable t. By the same reasoning as in (1) the new spiral W must be invariant.

*(iv) Argue that the spirals in the previous part are the only invariant curves of $\mathcal{F}_\tau$.*

Suppose the transformation is a translation. Let $\mathcal{W}$(t) = Z(t) + v.

$$\mathcal{F}_\tau[W(t)] = \mathcal{F}_\tau[Z(t) + v] = (\mathrm{Z}(\tau+\mathrm{t}) + e^{a(\tau)}\, e^{\mathrm{ib}(\tau)}\mathrm{v}) \neq \mathcal{W}(\mathrm{t})$$

$\mathcal{W}$(t) will not map a translated spiral into itself because under $\mathcal{F}_\tau$, the translation itself becomes a spiral rather than a constant point.

Suppose the transformation is a reflection in the x axis. Let $\mathcal{W}(\mathrm{t}) = \overline{Z(t)}$ [i.e., Conjugate[Z(t)]].

$$\mathcal{F}_\tau[W(t)] = \mathcal{F}_\tau[\overline{Z(t)}] = e^{\mathrm{a}\tau}\, e^{\mathrm{ib}\tau}\, e^{\mathrm{at}}\, e^{-\mathrm{ibt}} = e^{a(\tau+t)} e^{\mathrm{ib}(\tau-t)} \neq \mathcal{W}(\mathrm{t} + \tau)$$

$\mathcal{F}_\tau[W(t)]$ will not map a reflected spiral into itself.

Since neither translations of Z(t) nor reflections of Z(t) in the x-axis are invariant under $\mathcal{F}_\tau$, it seems likely that Z(t) under rotation is the only transformed curve that remains invariant under $\mathcal{F}_\tau$. Therefore only the original spiral and the rotated spiral constitute the invariant curves of $\mathcal{F}_\tau$.