Needham, Chapter 1, Exercise 26, p. 49
Palmer, February 4, 2020, 8:40 am PST

26 (i) By considering (a + ib)(cosθ + i sinθ), show that

$$b\cos\theta + a\sin\theta = \sqrt{a^2 + b^2}\ \sin\left[\theta + \tan^{-1}(b/a)\right]$$

(ii) Use this result to prove (14) by the method of induction.

Expand the product on the LHS, which is an equation for the spiral:

$$(a + ib)(\cos\theta + i\sin\theta) = a\cos\theta + i\,a\sin\theta + i\,b\cos\theta - b\sin\theta$$

$$= (a\cos\theta - b\sin\theta) + i\,(b\cos\theta + a\sin\theta) \qquad (1)$$

Rewrite the product on the LHS in polar format:

$$(a + ib)(\cos\theta + i\sin\theta) = \sqrt{a^2 + b^2}\ e^{i\tan^{-1}(b/a)}\ e^{i\theta}$$

$$= \sqrt{a^2 + b^2}\ e^{i\left(\theta + \tan^{-1}(b/a)\right)}$$

Rewrite the RHS using Euler's equation:

$$= \sqrt{a^2 + b^2}\ [\cos\ (\Theta + \ \mathtt{tan^{-1}(b\ /\ a)}\ )\ + i\sin\ (\Theta + \ \mathtt{tan^{-1}\ (b\ /\ a)}\ ] \qquad (2)$$

Set the imaginary parts of (1) and (2) equal, remembering that $r = \sqrt{a^2 + b^2}$ is a coefficient of both the real and imaginary parts of (2).

$$b\cos\theta + a\sin\theta = \sqrt{a^2 + b^2}\ \sin\ (\Theta + \ \mathtt{tan^{-1}(b\ /\ a)}\ ) \qquad (3)$$

This is what we wanted.

(ii) Use this result to prove (14) by the method of induction.

$$(14) \qquad \frac{d^n}{dt^n}\left[e^{at}\sin bt\right] = \left(a^2 + b^2\right)^{\frac{n}{2}}\ e^{at}\sin\left[bt + n\tan^{-1}(b/a)\right]$$

Comparison of (14) to (4) shows that one can let $\theta = bt$. Note that

$$\frac{d}{dt}\left[e^{at}\sin bt\right] = a\,e^{at}\sin bt + e^{at}\,b\cos bt = e^{at}\,(a\sin bt + b\cos bt)$$

Multiply both sides of (3) by e^{at} and substitute $\frac{d}{dt}[e^{at} \sin bt]$ in the LHS and bt for θ on the RHS. The factor e^{at} is an expansion as shown in [16], p. 21. Rewrite the square root with an exponent as preparation for the proof by induction.

$$e^{at}\, b \cos bt + e^{at}\, a \sin bt = \sqrt{a^2 + b^2}\ e^{at} \sin\left(\theta + \texttt{tan}^{-1}\left(\texttt{b} / \texttt{a}\right)\right)$$

$$\frac{d}{dt}[e^{at} \sin bt] = \left(a^2 + b^2\right)^{\frac{1}{2}} e^{at} \sin\left[bt + \tan^{-1}(b/a)\right] \tag{4}$$

This is the basis of induction, which we have proven with algebra. We can see from (3) that the derivative contributes a factor of $\left(a^2 + b^2\right)^{1/2}$ and it adds the angle $\tan^{-1}(b/a)$ to the angle bt. By comparison with (14), equation (4) appears to be the first in a series n = 1...k. We rewrite (14) for a particular n = **p** and find that a valid (parallel) expression using (**p** + 1) follows.

$$\frac{d^n}{dt^n}[e^{at} \sin bt] = \left(a^2 + b^2\right)^{\frac{p}{2}} e^{at} \sin\left[bt + \boldsymbol{p} \tan^{-1}(b/a)\right] \tag{5}$$

$$\frac{d^{n+1}}{dt^{n+1}}[e^{at} \sin bt] = \left(\left(a^2 + b^2\right)^{\frac{p}{2}} + \left(a^2 + b^2\right)^{\frac{1}{2}}\right) e^{at} \sin\left[bt + \boldsymbol{p} \tan^{-1}(b/a) + \tan^{-1}(b/a)\right]$$

$$= \left(a^2 + b^2\right)^{\frac{p+1}{2}} e^{at} \sin\left[bt + (\boldsymbol{p} + 1) \tan^{-1}(b/a)\right]$$

This is the expected outcome for the (**n**+1) derivative, and it also works for the first derivative, where **p** = 1. Therefore, (14) is proven for all natural numbers n.