Needham, Chapter 1, Exercise 24, p. 47.
Palmer, January 21, 2020, 8:10 pm PST

Out[395]=

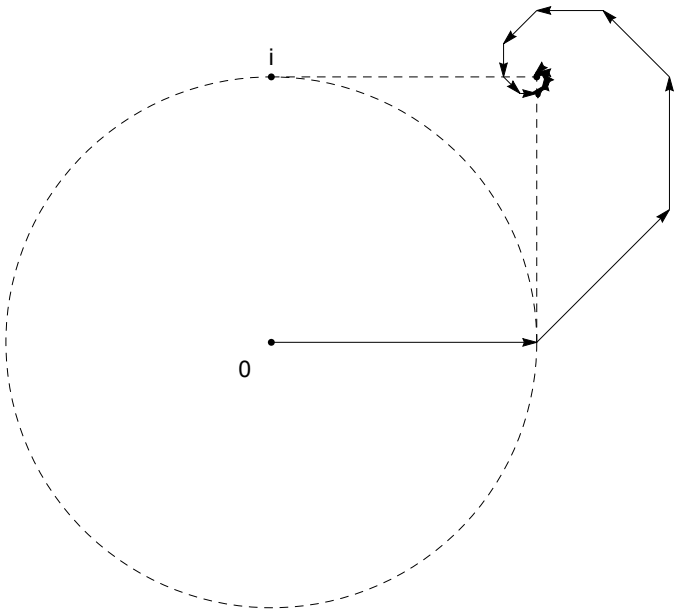


Figure 1. $(z^n-1)/(z-1)$
z = (1/2) (1 + i), n = 1..15

24 From (15), we know that

$$1 + z + z^2 + \cdots + z^{n-1} = \frac{z^n-1}{z-1}$$

(i) In what region of ℂ must z lie in order that the infinite series $1 + z + z^2 + \cdots$ converges.

Rewrite the RHS as $\frac{r^n e^{in\theta}-1}{re^{i\theta}-1}$. Then it is clear that the series can only converge if r = |x+iy| = $\sqrt{x^2 + y^2} < 1$, which is the description of the unit circle, so z must lie inside the unit circle.

(ii) If z lies in this region, to which point in the plane does the infinite series converge?

Given that z lies inside the unit circle, $\underset{n\to\infty}{\text{Lim}}\, z^n = 0$, and the series converges to

$$\underset{n\to\infty}{\text{Lim}}\, \frac{z^n-1}{z-1} = \frac{-1}{z-1}$$

If z = 0, the series equals 1, which is the first term or coefficient. If z = 1, the series is undefined and therefore cannot converge. For z = (1/2)(1+i), the series converges to (1+i).

(iii) In the spirit of figure [9], draw a large, accurate picture of the infinite series in the case $z = \frac{1}{2}(1 + i)$*, and check that it does indeed converge to the point predicted by part (ii).*

See Figure 1. Let S be the limit of the series at n = ∞.

For $z = \frac{1}{2}(1 + i)$,

$$S = \frac{-1}{z-1} = \frac{-1}{\frac{1}{2}(1+i)-1} = \frac{-2}{(1+i)-2} = \frac{-2}{i-1} = \frac{2}{1-i} = \frac{2(1+i)}{(1-i)(1+i)} = \frac{2(1+i)}{2} = (1+i)$$

This is the point of convergence shown in Figure 1.