Needham, Chapter 1, Exercise N, p. 47.
Palmer, January 19, 2020, 9:30 pm PST

Out[54]=

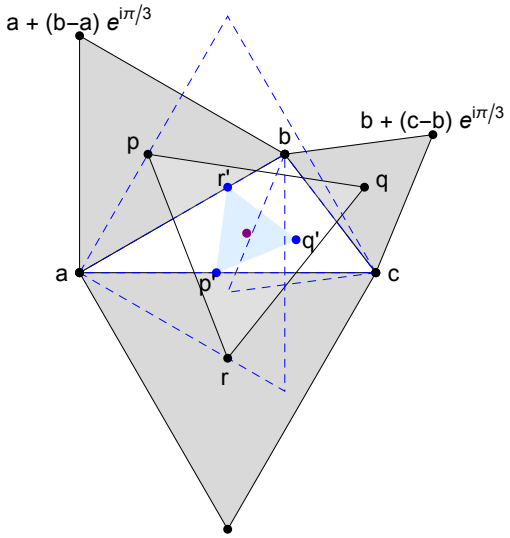


Figure 1. Exterior and interior equilateral triangles
with vertices p, q, r and p', q', r'
at centroids of equilateral triangles
on sides of arbitrary triangle a, b, c

In[55]:=

23 Draw an arbitrary triangle, and on each side draw an equilateral triangle lying outside the given triangle. What do you suspect is special about the new triangle formed by joining the centroids (cf. Ex. 19) of the equilateral triangles? Use complex algebra to prove that you are right. What happens if the equilateral triangles are instead drawn on the inside of the given triangle?

What do you suspect is special about the new triangle formed by joining the centroids (cf. Ex. 19) of the quilateral triangles?

Based on Exercise 21, which revealed orthogonal angles between segments joining the vertices, one would expect an equilateral triangle. Inspection of Figure 1 suggests that is correct. If so, it should be sufficient to prove that (q-p) = $(r - p)\, e^{i\pi/3}$, where p, q, and r are centroids of the equilateral triangles as shown in Figure 1. The letters a, b, and c indicate complex numbers (points), so a = 0. This is a different scheme from that used in [12a], where the letters indicated vectors.

The vertices of the equilateral triangles, named for their centroids:

P = {a, a + (b-a) $e^{i\pi/3}$, b}

Q = {b, b + (c-b) $e^{i\pi/3}$, c}

$$R = \{c, c + (a-c)\, e^{i\pi/3}, a\}$$

The centroids of the equilateral triangles:

$$p = (1/3)\,(a + (a+ (b-a)\, e^{i\pi/3}) + b)$$

$$= (1/3)\,(2a + b + (b-a)\, e^{i\pi/3})$$

$$= (1/3)\,(2a + b + (b-a)\,(1/2)\,(1+i\sqrt{3}))\qquad [e^{i\pi/3} = (1/2)\,(1+i\sqrt{3})]$$

$$= (1/3)\,(2a + b + (1/2)\,(1+i\sqrt{3})\, b - (1/2)\,(1+i\sqrt{3})\, a)$$

$$= (1/3)\,((3/2)a + (3/2)\, b + (i\sqrt{3}/2)b - (i\sqrt{3}/2)a$$

$$= (1/3)\,[(3/2 - (i\sqrt{3}/2))\, a + (3/2 + i\sqrt{3}/2)\, b]$$

$$q = (1/3)\,(b + (b + (c-b)\, e^{i\pi/3}) + c)$$

$$= (1/3)\,(2b + c + (c-b)\, e^{i\pi/3})$$

$$= (1/3)\,(2b + c + (c-b)\,(1/2)\,(1+i\sqrt{3}))$$

$$= (1/3)\,(2b + c + (1/2)\,(1+i\sqrt{3})\, c + (1/2)\,(1+i\sqrt{3})\,(-b))$$

$$= (1/3)\,(2 - 1/2 - (1/2)\, i\sqrt{3})\, b + (1 + 1/2 + (1/2) i\sqrt{3}\, c$$

$$= (1/3)\,[((3/2) - (1/2)\, i\sqrt{3})\, b + (3/2 + (1/2)\, i\sqrt{3})\, c]$$

$$r = (1/3)\,(c + c + (a-c)\, e^{i\pi/3} + a)$$

$$= (1/3)\,(a + 2c + (a-c)\, e^{i\pi/3})$$

$$= (1/3)\,(a + 2c + (a-c)\,(1/2)\,(1+i\sqrt{3}))$$

$$= (1/3)\,(1 + 1/2 + i\sqrt{3}/2)\, a + (2 - 1/2 - i\sqrt{3}/2)\, c$$

$$= (1/3)\,[\,(3/2 + i\sqrt{3}/2)\, a + (3/2 - i\sqrt{3}/2)\, c\,]$$

Show that

$$(q-p) - (r-p)\, e^{i\pi/3} = (q-p) - (r-p)\,(1/2 + i\sqrt{3}/2) = 0$$

$$(q-p) = (1/3)\,[((3/2) - (1/2)\, i\sqrt{3})\, b + (3/2 + (1/2)\, i\sqrt{3})\, c] - [(3/2 - (i\sqrt{3}/2))\, a + (3/2 + i\sqrt{3}/2)\, b]]$$

$$= (1/3)\,(\,(-3/2 + (i\sqrt{3}/2))\, a + ((3/2) - (1/2)\, i\sqrt{3} - 3/2 - i\sqrt{3}/2))\, b + (3/2 + (1/2)\, i\sqrt{3})\, c$$

$$= (1/3)\ (\ (-3/2 + (i\sqrt{3}\ /2)\ a - i\ \sqrt{3}\ b\ +\ (3/2 + (1/2)\ i\sqrt{3}\ )\ c$$

$$(r\text{-}p) = (1/3)\ [[\ (3/2 + i\sqrt{3}\ /2)\ a + (3/2 - i\sqrt{3}\ /2)\ c\ ] -\ \ [(3/2 -\ (i\sqrt{3}\ /2))\ a + (3/2 + i\ \sqrt{3}\ /2)\ b]]$$

$$=\ (1/3)\ [\ i\sqrt{3}\ \ a - (3/2 + i\ \sqrt{3}\ /2)\ b + (3/2 - i\sqrt{3}\ /2)\ c\ ]$$

$$(r\text{-}p)\ (1/2 + i\sqrt{3}\ /2) = (1/3)\ [\ i\sqrt{3}\ \ a - (3/2 + i\ \sqrt{3}\ /2)\ b + (3/2 - i\sqrt{3}\ /2)\ c\ ]\ (1/2)\ (1 + i\ \sqrt{3}\ )$$

$$=\ (1/3)\ [\ i\sqrt{3}\ \Big/ 2\ a - (3/4 + i\ \sqrt{3}\ /4)\ b + (3/4 - i\sqrt{3}\ /4)\ c\ ] +\ i\sqrt{3}\ /2\ [\ i\sqrt{3}\ \ a - (3/2 + i\ \sqrt{3}\ /2)\ b + (3/2 - i\sqrt{3}\ /2)\ c\ ]$$

$$= (1/3)\ [\ i\sqrt{3}\ \Big/ 2\ a - (3/4 +\ i\ \sqrt{3}\ /4)\ b + (3/4 - i\sqrt{3}\ /4)\ c)\ \ - (3/2)\ a - (i3\ \sqrt{3}\ /4) + 3/4)\ b\ +\ (i3\ \sqrt{3}\ /4 + 3/4)\ c\ ]$$

$$= (1/3)\ [(-3/2) + i\sqrt{3}\ \Big/ 2\Big)\ a\ + (-3/4 + 3/4 -\ i\ \sqrt{3}\ /4 - i3\ \sqrt{3}\ /4)\ b + (3/4 + 3/4 - i\sqrt{3}\ /4 + i3\ \sqrt{3}\ /4)\ c$$

$$=\ (1/3)\ [(-3/2) + i\sqrt{3}\ \Big/ 2\Big)\ a -\ i\ \sqrt{3}\ b\ + (3/2 +\ (i\sqrt{3}\ /2)\ c]$$

$$(q\text{-}p) - (r\text{-}p)\ e^{i\pi/3} =\ \ (1/3)[\ [\ \ (\ (-3/2 + (i\sqrt{3}\ /2)\ a - i\ \sqrt{3}\ \ b\ +\ \ (3/2 + (1/2)\ i\sqrt{3}\ )\ c\ ]$$

$$- [(-3/2) + i\sqrt{3}\ \Big/ 2\Big)\ a -\ i\ \sqrt{3}\ b\ + (3/2 +\ (i\sqrt{3}\ /2)\ c]$$

$$=\ 0$$

This is what we wanted. Comparison with Vasco's answer shows that the method described by Needham shown in [12a] and discussed on pp. 16 and 17 requires less algebra and is therefore preferred if one must do the algebra manually. However, if one is concerned with convenience, this method can be tested with a short computer program. The following is a four line *Mathematica* program which verified the equation $(q\text{-}p) = (r\text{-}p)\ e^{i\pi/3}$.

```
p = (1 / 3) (a + a + (b - a) Exp[I π / 3] + b);
q = (1 / 3) (b + b + (c - b) Exp[I π / 3] + c);
r = (1 / 3) (c + c + (a - c) Exp[I π / 3] + a);
result = (3 (q - p) == 3 (r - p) Exp[I π / 3]) // Simplify;
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p: $\frac{1}{3}\left(2\,a + b + (b - a)\,e^{\frac{i\pi}{3}}\right)$

q: $\frac{1}{3}\left(2\,b + c + (c - b)\,e^{\frac{i\pi}{3}}\right)$

r: $\frac{1}{3}\left(a + 2\,c + (a - c)\,e^{\frac{i\pi}{3}}\right)$

q−p: $\frac{1}{3}\left(-2\,a - b - (b - a)\,e^{\frac{i\pi}{3}}\right) + \frac{1}{3}\left(2\,b + c + (c - b)\,e^{\frac{i\pi}{3}}\right)$

(r−p) e^(iπ/3): $e^{\frac{i\pi}{3}}\left(\frac{1}{3}\left(-2\,a - b - (b - a)\,e^{\frac{i\pi}{3}}\right) + \frac{1}{3}\left(a + 2\,c + (a - c)\,e^{\frac{i\pi}{3}}\right)\right)$

(q−p == (r−p) e^(iπ/3): True

The equation is true.

What happens if the equilateral triangles are instead drawn on the inside of the given triangle?

The inside equilateral triangles are indicated with blue dashed lines. The triangle with vertices at the inside centroids is a smaller equilateral triangle, such as the light blue centred triangle in Figure 1. This is because the new centroids are reflections about their corresponding side of the arbitrary triangle, and the movement of each is proportionate to the size of the originating side of the arbitrary triangle. The reflected centroids are indexed to their reflections with p', q', and r'. The (purple) centroid of the inside triangle is identical to the centroids of the arbitrary triangle and the exterior equilateral triangle.