Needham, Chapter 1, Exercise N, p. 47.
Palmer, January 16, 2020, 5:50 pm PST

Out[4896]=

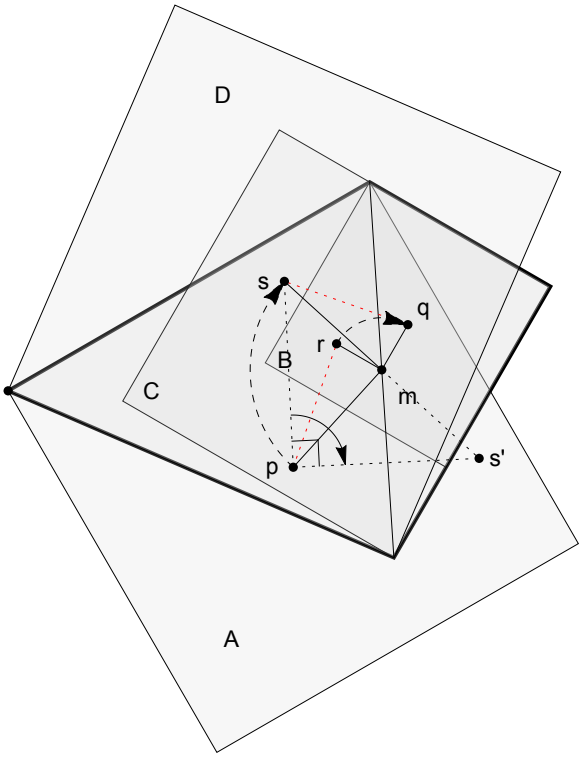


Figure 1. Squares on interior of arbitrary quadrilateral

In[73]:=

22 Will the result in [12a] survive if the squares are instead constructed on the inside of the quadrilateral.

In Figure 1, A, B, C, and D refer to the squares based on sides 2a, 2b, 2c, and 2d, respectively, as in [12a]. The centre points p, q, r, s are not the same as the ones labeled p, q, r, s in [12a], but they are symmetrical with those across their common quadrilateral sides. The following is the proof from algebra based on the method presented on pp. 16 and 17. Where the proof on pp. 16 and 17 uses (s-q) + i(r-p) = 0, Figure 1 seems to call for (s-q) + i(p-r) = 0. Either equation should yield the same result with the new configuration as both depend on the closure of (a+b+c+d). A similar result to [12a] survives up to the change in orientation of (r-p) to (p-r).

Show that (s-q) + i(p-r) = 0

p = a - ia
q = 2a + b - ib
r = 2a + 2b + c - ic
s = 2a + 2b + 2c + d - id

p-r = a - ia - (2a + 2b + c - ic)
= -a - 2b - c + ic -ia
= -a -2b - c + i (c-a)
i (p-r) = (-c + a) - i(a+2b+c)

s-q = 2a + 2b + 2c + d - id - (2a + b - ib)
= b + 2c + d - id + ib
= b + 2c + d + i (b-d)

(s-q) + i(p-r) = (b + 2c + d + i (b-d)) + ((-c + a) - i(a+2b+c)))
= a - b - c + d + i (c - a - b - c - d)
= - a - b - c - d + i (-a - b - c - d) = -0+0 = 0

showing that pr ⊥ sq and |pr| = |sq|.

We go beyond the question here and show that sm and pm are also perpendicular and of equal length. Following the reasoning in the last equation on p. 19 and inspection of Figure 1, we can write

$$s' = \left(\mathcal{R}_p^{-\pi/2} \circ \mathcal{R}_s^{-\pi/2}\right)(s) = \mathcal{R}_p^{-\pi/2}(s)$$

showing that the triangle sps' is isosceles and has a right angle at p, so sm and pm are perpendicular and of equal length. Note that the rotation is $-\pi/2$ rather than $\pi/2$. Therefore our conjectures from inspection that the result in [12b] and [15a] from exercise 21 survives up to a change in orientation are supported.