Needham, Chapter 1, Exercise 21, p. 47.
Palmer, January 16, 2020, 10:45 am PST

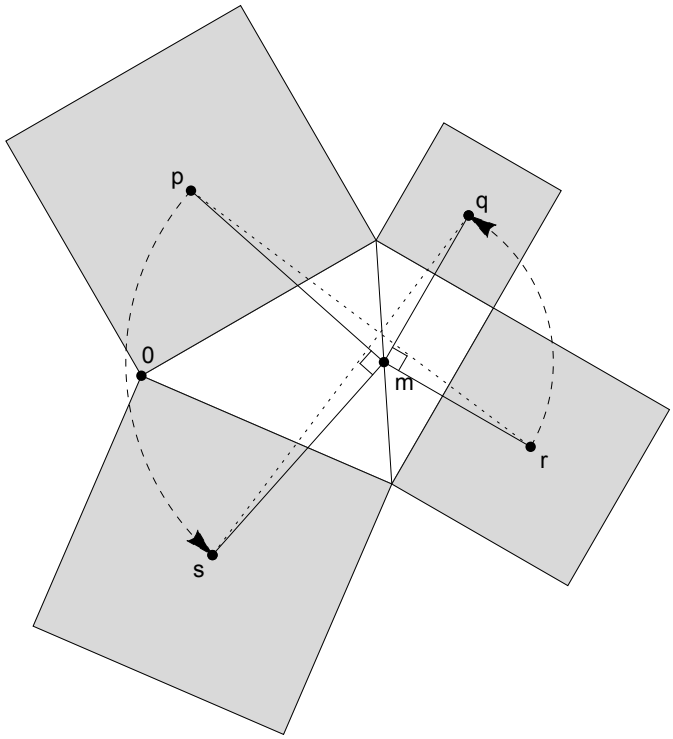


Figure 1. Squares on an arbitrary quadrilateral

21 Make a copy of [12a], draw in the diagonal of the quadrilateral shown in [12b] and mark its midpoint m. As in [12b], draw the line-segments connecting m to p, q, r, and s. According to the result in [12b], what happens to p and to r under a rotation of π/2 about m? So what happens to the line segment pr? Deduce the result shown in [12a].

Figure 1 is like figure [12b] expanded to include the q and r squares. The lines from m to p, q, r, and s are foregrounded in contrast to figure [12a], which foregrounded the lines pr and qs and in contrast to [12b], which reduced the profiles of the lines pm and sm with dashed lines. The arrangement in Figure 1 seems more appropriate to the problem statement.

According to the result in [12b], what happens to p and to r under a rotation of π/2 about m?

The result in [12b] is found at the end of subsection 3, p. 19:

Thus the triangle sps' is isosceles and has a right angle at p, so sm and pm are perpendicular and of equal length.

On early readings, I failed to see how [15a] on p. 20 could be asserted to be the same as [12b] on p.

16. Figure [12b] was obtained by a construction process involving four squares and an arbitrary quadrilateral. We know that pr ⊥ qs and |pr| = |qs| in [12a] because it is proven on pp. 16 and 17.

Then, the midpoint m of a diagonal was found. On p. 17, it is asserted, but not yet proven, that pm = i sm, and so, it must also be the case that qm = i rm. The proof that pm = i sm is found on p. 19, where the four-line quote in the centre of the page provides the basis for the algebra of the proof.* The proof refers the reader to the construction in [12b] and at least implicitly begins with that construction in [15a]. Hence, we know that m is the midpoint of the diagonal in both figures. These facts provide the basis for answering the first question, what happens to p and r under a rotation of $\pi/2$ about m.

Since pm and sm are perpendicular and of equal length, it is obvious that p must go to s and r must go to q.

So what happens to the line-segment pr? Deduce the result shown in [12a].

Since p has gone to s and r has gone to q, the line segment pr must replace the line segment sq. The line segment sq is similarly rotated by $\pi/2$ about m to replace the line segment pr. Because the two lines replace one another after rotation by $\pi/2$ about m, they must be perpendicular.

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*FOOTNOTE

At first I was unsure what Needham meant by "the result in [12b]". I thought he might mean that pm is perpendicular to sm and |pm| = |sm|. I wrote this proof that pm is perpendicular to sm and |pm| = |sm|. Needham's proof is quite different.

Under a rotation of $\pi/2$ about m, p would fall on the extension of sm and r would fall on the extension of qm. In fact, p falls on s and r falls on q. If this is true, then q-m = i (r-m) and (q-m) - i(r-m) = 0. We show that r falls on q.

q-m = i (r-m)

q = 2a + (1+i)b

r = 2a + 2b + (1+i)c;

m = ((2a +2b + 2c) - 2a)/2 + 2a
  = (2a + b+c)

q-m = 2a + (1+i)b - (2a + b+c)
  = ib - c

r-m = 2a + 2b + (1+i) c -(2a + b + c)
  = b + ic

$$ib - c = i\,(b + ic)$$

$$ib - c - ib + c = 0$$

The equation is true.

We can write the same test for the rotation of p to s. If p falls on s, then s-m = i (p-m) and (s-m) - i(p-m) = 0

$$p = a + ia$$

$$s = 2a + 2b + 2c + (1+i)d$$

$$p-m = (a + ia) - (2a + b+c)$$

$$= -a - b - c + ia$$

$$s-m = (2a + 2b + 2c + (1+i)d) - (2a + b+c)$$

$$= b + c + d + id$$

$$b + c + d + id = i\,(-a - b - c + ia)$$

$$a + b + c + d = i\,(-a -b -c - d)$$

$$a + b + c + d = -i\,(a + b + c + d)$$

Again, the equation is 0 because (a+b+c+d) = 0.