Needham, Chapter 1, Exercise 20, p. 47.
Palmer, January 18, 2020, 10:35 am PST

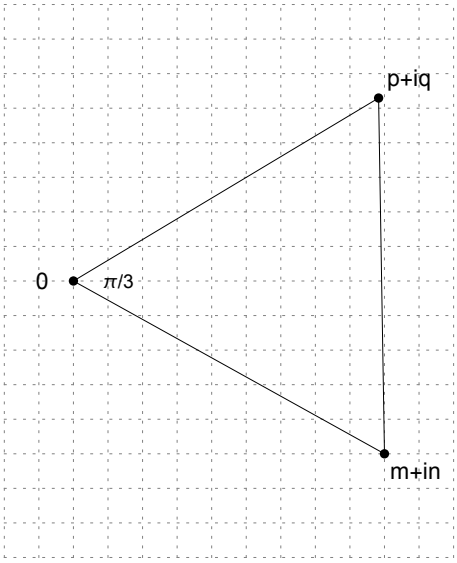


Figure 1. Equilateral triangle doesn't fit Gaussian grid

20 Gaussian integers are complex numbers of the form m+in, where m and n are integers—they are the grid points in [1]. Show that it is impossible to draw an equilateral triangle such that all three vertices are Gaussian integers. [Hints: You may assume that one of the vertices is at the origin; try a proof by contradiction; if a triangle is equilateral, you can rotate one side into another; remember that $\sqrt{3}$ *is irrational.]*

My answer follows the approach of Vasco but does not duplicate it in every detail. Referring to the equilateral triangle in Figure 1, we know that ∠[(m+in)-0-(p+iq)] is $\pi/3$. By rotating (m+in) by $\pi/3$ about the origin, we rotate the side (m+in) into the side (p+iq). In Figure 1, (m+in) was chosen to be a Gaussian integer. We will now assume that (p+iq) is also a Gaussian integer. Then every quotient involving p or q should also be rational.

$$p+iq = (m+in)\, e^{i\pi/3}$$

$$= (m+in)\,(\cos(\pi/3) + i \sin(\pi/3)$$

$$= (m+in)\left(\frac{1}{2} + i\,\frac{\sqrt{3}}{2}\right)$$

$$= \left(\frac{1}{2}m + i\ \frac{\sqrt{3}}{2}m\right) + \left(\frac{1}{2}(in) + i\ \frac{\sqrt{3}}{2}(in)\right)$$

$$= \left(\frac{1}{2}m - \frac{\sqrt{3}}{2}n\right) + i\left(\frac{\sqrt{3}}{2}m + \frac{1}{2}n\right)$$

Setting real and imaginary parts of the two sides equal, we can write

$$p = \left(\frac{1}{2}m - \frac{\sqrt{3}}{2}n\right) \qquad (1)$$

$$q = \frac{\sqrt{3}}{2}m + \frac{1}{2}n \qquad (2)$$

Equation (1) implies that $n = \frac{m-2p}{\sqrt{3}}$. Since m and p must both be integers by the original assumption, n can not be an integer because $\sqrt{3}$ is irrational. This is a contradiction with the original premise that (m + in) and (p + iq) are Gaussian integers.

Equation (2) implies that $m = \frac{2q-n}{\sqrt{3}}$. Since q and n must both be integers by the original assumption, m cannot be an integer. This is again a contradiction with the original premise that (m + in) and (p + iq) are Gaussian integers.

It follows that it is impossible to draw an equilateral triangle such that all three vertices are Gaussian integers. Now we can also appreciate the small deviation of (p+iq) from a grid point in Figure 1 as the triangle was constructed to be equilateral.