Needham, Chapter 1, Exercise 2, p. 45.
G. Palmer, December 19, 2019, 6:50 pm, PST

2 In order to solve the cubic equation $x^3 = 3px + 2q$, do the following:

(i) make the inspired substitution $x = s + t$, and deduce that x solves the cubic if $st = p$ and $s^3 + t^3 = 2q$.

What would inspire Cardano to let x = s + t? Perhaps the answer will become clear.

$$x^3 = 3px+2q = 3p(s+t) + 2q$$

$$(s + t)^3 = 3p(s+t) + 2q$$

$$s^3 + 3s^2 t + 3st^2 + t^3 = 3p(s+t) + 2q$$

$$\left(3s^2 t + 3st^2\right) + (s^3 + t^3) = 3p(s+t) + 2q$$

Now it is not hard to see that $\left(3s^2 t + 3st^2\right) = 3p(s+t)$ if p = st. Then we can let $2q = (s^3 + t^3)$.

$$3st\,(s + t) + (s^3 + t^3) = 3p(s+t) + 2q$$

This equation is true if p = st and $2q = (s^3 + t^3)$, so x does solve the cubic if x = s+t. Perhaps when trying to solve a difficult equation for x, it is a good idea to let x equal a sum of two or more numbers and see what patterns emerge.

(ii) Eliminate t between these two equations, thereby obtaining a quadratic equation in s^3.

$$t = \frac{p}{s}$$

Substitute for t in the second equation $2q = (s^3 + t^3)$.

$$2q = s^3 + \left(\frac{p}{s}\right)^3$$

$$s^3 + \left(\frac{p}{s}\right)^3 - 2q = 0$$

$$\left(s^3\right)^2 - 2\,qs^3 + p^3 = 0$$

(iii) Solve this quadratic to obtain the two possible values of s^3. By symmetry, what are the possible values of t^3.

$$s^3 = \frac{2q \pm \sqrt{4q^2 - 4\left(1 \cdot p^3\right)}}{2}$$

$$= q \pm \sqrt{q^2 - p^3}$$

The possible values of t^3 are the same as those of s^3.

(iv) Given that we know that $s^3 + t^3 = 2q$, deduce the formula (4).

Since $s^3 + t^3 = 2q$, we know that the square root term of s^3 is opposite in sign to that of t^3, for otherwise we would have

$$2q = 2q + 2\sqrt{q^2 - p^3}$$

or

$$2q = 2q - 2\sqrt{q^2 - p^3}$$

which would be true only if $q^2 = p^3$, but not true in general. Then, since it is given that x = s + t and s must differ from t, (4) is the only possibility.