Needham, Chapter 1, Exercise 19, p. 47.
Palmer, January 13, 2020, 1 pm PST

Out[1022]=

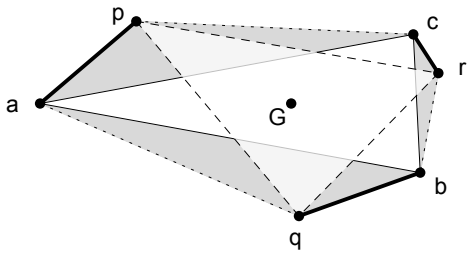


Figure 1. $\tilde{G} = G = \frac{1}{3}(a+b+c)$

19 The "centroid" G of a triangle T is the intersection of its medians. If the vertices are the complex numbers a, b, and c, then you may assume that $G = \frac{1}{3}(a + b + c)$ *(Figure 1). On the sides of T we have constructed three similar triangles [dotted] of arbitrary shape, so producing a new triangle [dashed] with vertices p, q, r. Using complex algebra, show that the centroid of the new triangle is in exactly the same place as the centroid of the old triangle!*

A convenient way to generate the three similar triangles (Figure 1, gray) and find the vertices of the new triangle at the same time is to multiply each side of the original triangle by the same complex number $Re^{i\Theta}$, proceeding in order around the three original vertices (Figure 1, thick lines, p-a, q-b, r-c). Multiplying all sides by the same complex number assures that the length of every new side of the new vertex (say sides [ap], [bq], and [cr] of vertices p, q, and r) has the same ratio to a corresponding side of the original ([ac], [ba], [cb]), and it assures that ∠pac = ∠qba = ∠rcb. With two sides in the same ratio, and one equal angle, the three new triangles are similar. The new triangle with centroid G (yet to be proven) is shown in dashed lines superimposed on the old and the similar triangles.

Let $\tilde{G}$ be the centroid of the new triangle. Show that $\tilde{G} = G$.

$$p = a + Re^{i\Theta}(c-a)$$
$$q = b + Re^{i\Theta}(a-b)$$
$$r = c + Re^{i\Theta}(b-c)$$

Then

$$\tilde{G} = \frac{1}{3}(p+q+r)$$

$$= \frac{1}{3}(a+b+c) + \frac{1}{3}(Re^{i\Theta}(c-a) + Re^{i\Theta}(a-b) + Re^{i\Theta}(b-c))$$

$$= G + \frac{1}{3}Re^{i\Theta}(c - a + a - b + b - c) = G$$

This is what we wanted.