Needham, Chapter 1, Exercise 18, p. 47.
Palmer, January 11, 2020, 7 pm PST

18 Prove that

$$e^{i\theta} + e^{i\phi} = 2\cos\left[\frac{\theta-\phi}{2}\right] e^{\frac{i(\theta+\phi)}{2}} \text{ and } e^{i\theta} - e^{i\phi} = 2\,i\sin\left[\frac{\theta-\phi}{2}\right] e^{\frac{i(\theta+\phi)}{2}}$$

(i) by calculation, and (ii) with a picture.

(i) by calculation,

Divide through by $e^{\frac{i(\theta+\phi)}{2}}$.

$$\frac{e^{i\theta}+e^{i\phi}}{e^{\frac{i(\theta+\phi)}{2}}} = 2\cos\left[\frac{\theta-\phi}{2}\right]$$

$$e^{\frac{i(\theta-\phi)}{2}} + e^{\frac{-i(\theta-\phi)}{2}} = 2\cos\left[\frac{\theta-\phi}{2}\right]$$

This satisfies (12). Done.

Do the same for the second equation.

$$\frac{e^{i\theta}-e^{i\phi}}{e^{\frac{i(\theta+\phi)}{2}}} = 2\,i\sin\left[\frac{\theta-\phi}{2}\right]$$

$$e^{\frac{i(\theta-\phi)}{2}} - e^{\frac{-i(\theta-\phi)}{2}} = 2\,i\sin\left[\frac{\theta-\phi}{2}\right]$$

This also satisfies (12).

Out[3355]=

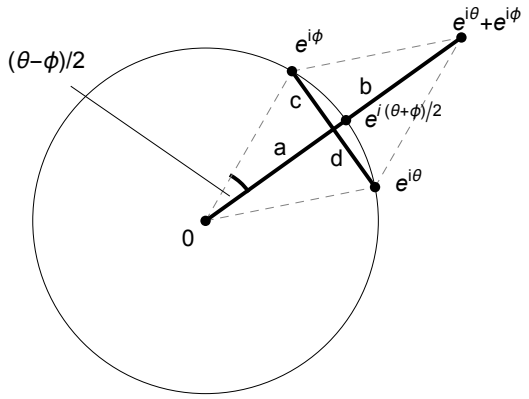


Figure 1. Angle sums, exponential and trig

(ii) with a picture.

In Figure 1, a = b = $cos\left[\frac{\theta-\phi}{2}\right]$. One can see that these real values have been rotated by arg($e^{i\theta} + e^{i\phi}$), which we have learned is equivalent to (θ+ϕ)/2 because the unit vectors permit us to plot a rhombus that is bisected by the vectors $\left(e^{i\theta} + e^{i\phi}\right)$ and $\left(e^{i\theta} - e^{i\phi}\right)$. By rotating and doubling the cosine we obtain $e^{i\theta} + e^{i\phi}$, verifying that $e^{i\theta} + e^{i\phi} = 2\, cos\left[\frac{\theta-\phi}{2}\right] e^{\frac{i(\theta+\phi)}{2}}$.

We can also see that c = d = $sin\left[\frac{\theta-\phi}{2}\right]$, which is also rotated by arg($e^{i\theta} + e^{i\phi}$) = (θ+ϕ)/2. By rotating and doubling the sine, and multiplying by i because it is perpendicular to the cosine, we obtain the vector $\left(e^{i\theta} - e^{i\phi}\right)$, verifying that $e^{i\theta} - e^{i\phi} = 2\, i\, sin\left[\frac{\theta-\phi}{2}\right] e^{\frac{i(\theta+\phi)}{2}}$.