Needham, Chapter 1, Exercise 17, p. 47.
Palmer, January 11, 2020, 3 pm PST

Out[1605]=

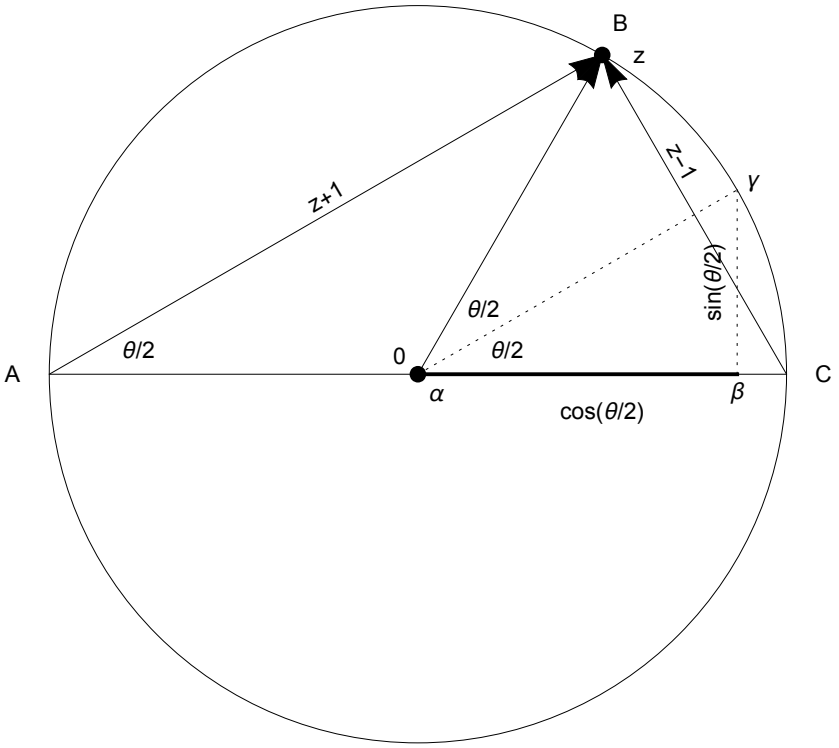


Figure 1. $(z-1) = (i \tan \frac{\theta}{2})\,(z+1)$

17 If $z = e^{i\theta} \neq -1$, then $(z-1) = (i \tan \frac{\theta}{2})\,(z+1)$. Prove this (i) by calculation, (ii) with a picture.

Let $C = \cos\theta$, $S = \sin\theta$.

$z - 1 = \cos\theta + i \sin\theta - 1 = C+iS-1$
$z+1 = \cos\theta + i \sin\theta + 1 = C+iS+1$

$\tan(\theta/2) = \sin\theta/(1+\cos\theta)$ (trig identity)
$\tan(\theta/2) = S/(1+C)$

Rewrite the equation with substitutions for (z-1), (z+1), and $\tan(\theta/2)$.

$$C+iS-1 = [i\ S/(1+C)]\ (C+iS+1)$$

$$(C+iS-1)\ (1+C) = iS\ (C+iS+1)$$

$$C + C^2 + iS + iCS - 1 - C = iCS - S^2 + iS$$

$$C^2 + S^2 - 1 = 0 \qquad \text{by subtracting RHS from LHS}$$

Since $C^2 + S^2 = 1$, the equation is proven with algebra.

(ii) with a picture.

Figure 1 shows that $\triangle$ABC is similar to $\triangle\alpha\beta\gamma$. From this is follows that the vector quotient $(z-1)/(z+1) = i \sin(\theta/2)/\cos(\theta/2) = i \tan(\theta/2)$.