Needham, Chapter 1, Exercise 16, p. 47.
Palmer, January 12, 2020, 8:10 am PST

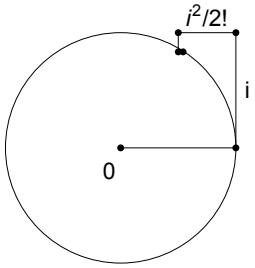


Figure 1. $e^{i\theta} = 1 + i\theta + \left(\frac{i\theta}{2!}\right)^2 + \left(\frac{i\theta}{3!}\right)^3 + \ldots$, θ=1

16 Starting from the origin, go one unit east, then the same length north, then (1/2) of the previous length west, then (1/3) of the previous length south, then (1/4) of the previous length east, and so on. What point does this spiral converge to?

See the discussion of Subsection 3, Power Series Argument, p. 12. If one goes one unit east, it is one unit on the positive x axis. "one unit north" is equivalent to i·1. Then (1/2) the previous length west would be -(1/2), equivalent to $i^2/2 = i^2/2!$. Then (1/3) of the previous length south would be -(1/3)i, equivalent to (1/3)(1 / 2) $i^3 = i^3 \,/\, 3!$, and so on. Vasco (Forum) has offered a better summation: *I think that Needham wants us to imagine someone walking east, north, west, south, east,... and to think of them turning through 90 degrees anticlockwise after each straight line walk, so that we can write each 'walk' as the previous walk multiplied by i to get the 90 degree turn and dividing by 1,2,3,... to get the appropriate length. This way we get the terms of the series automatically.*

The sequence described is an instance of the series

$$e^{i\theta} = 1 + i\theta + \left(\frac{i\theta}{2!}\right)^2 + \left(\frac{i\theta}{3!}\right)^3 + \ldots$$

where $\theta = 1$. The spiral converges to e^i, which is a point on the unit circle at the angle of 1 radian.