Needham, Chapter 1, Exercise 13, p. 47.
G. Palmer, January 9, 2020, 4 am PST

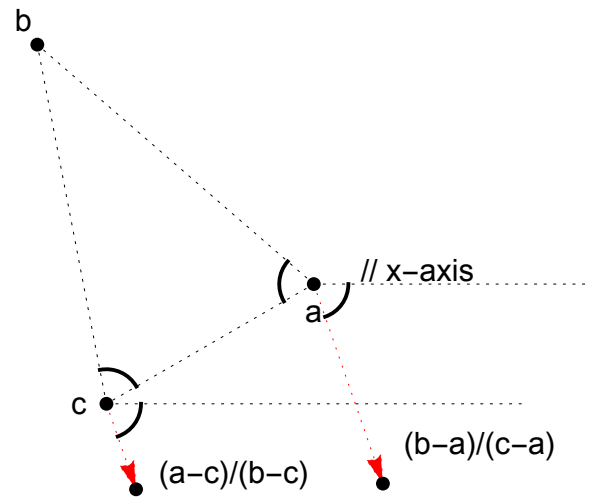


Figure 1. ? ∠(b–a)/(c–a) = ∠(a–c)/(b–c)

13 Find the geometric configuration of the points a, b, and c if

$$\left(\frac{b-a}{c-a}\right) = \left(\frac{a-c}{b-c}\right)$$

[Hint: Separately equate the lengths and angles of the two sides.]

Following the hint:

$$\arg\left(\frac{b-a}{c-a}\right) = \arg\left(\frac{a-c}{b-c}\right) \qquad (1)$$

The angles with the thick arc are equal, as in Figure 1. Since ∠bac = ∠bca, then two sides are also equal:

$$|b-a| = |b-c|$$

Now we can see that Figure 1 must be wrong because the red vector arrows have different lengths showing different ratios for the two sides of the equation.

Continuing with the hint:

$$\left|\frac{b-a}{c-a}\right| = \left|\frac{a-c}{b-c}\right| \qquad (2)$$

The LHS of the equation gives the ratio of side (b-a) to side (c-a) while the RHS gives the ratio of the side (a-c) to side (b-c) (Figure 1). We know that

|a-c| = |c-a|

and

|b-a| = |b-c|

which, after multiplying through by the denominators in (2), implies that the squares of |a-c|, |b-a|, and |b-c| are equal, showing that all three sides are equal and the triangle is equilateral. [A problem with the *Mathematica* editor prevented me from putting this in the form of equations.]

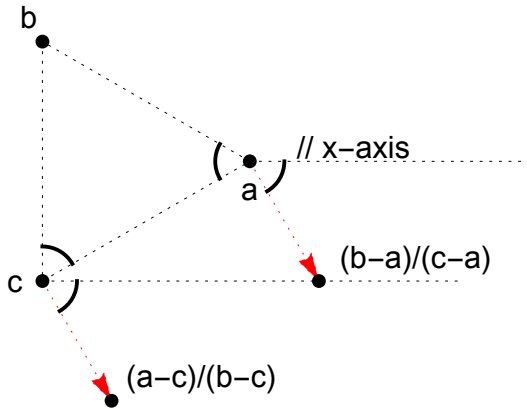


Figure 2. |(b−a)/(c−a)| = |(a−c)/(b−c)|