Needham, Chapter 1, Exercise 12, p. 46.
G. Palmer, January 9, 2020, 9 am PST

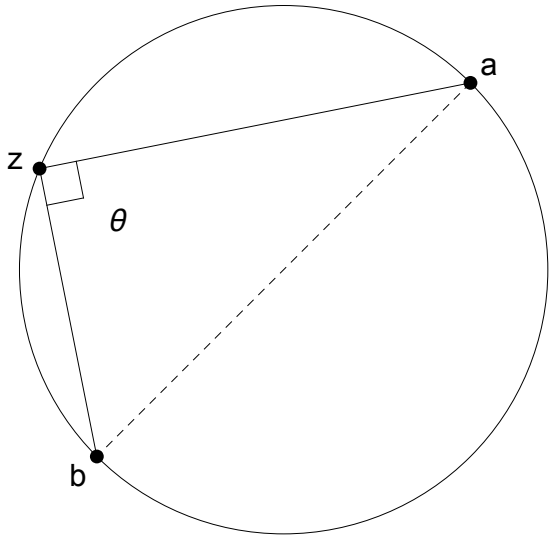


Figure 1. Re((z−1−i)/(z+1+i)) = 0

12 By using pictures, find the locus of z for each of the following equations:

$$\mathrm{Re}\left(\frac{z-1-i}{z+1+i}\right) = 0, \quad \text{and} \quad \mathrm{Im}\left(\frac{z-1-i}{z+1+i}\right) = 0$$

[Hints: What does Re(W) = 0 imply about the angle of W? Now use the previous exercise.]

Re(W) = 0 implies that ∠W = ±π/2. Using the previous exercise, a = (1+i) and b = -(1+i). Then

$$\arg\left(\frac{z-a}{z-b}\right) = \pm\pi/2$$

for all z on the circular arc subtended by the chord ab. Then from the angle ±π/2 we also know that the line segment ab is a diameter of C and that the locus of z is the whole circle minus point b (Figure 1). When z = b, the quotient is undefined. But one might easily resolve the problem by rotating C by an infinitesimal angle about its centre enabling z=b without creating a singularity.

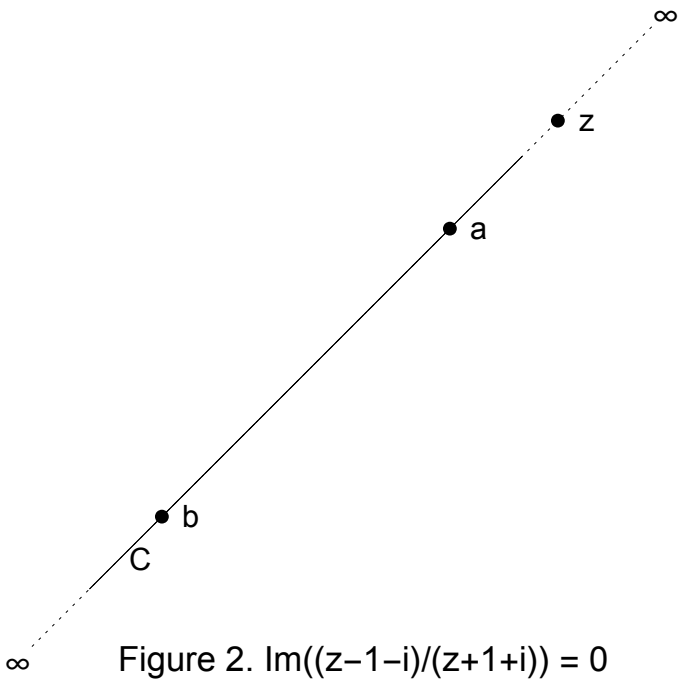


Figure 2. Im((z−1−i)/(z+1+i)) = 0

Im(W) = 0 implies that ∠W = 0 or π. Recalling from p. 36 and exercise 11 that

$$arg\left(\frac{z-a}{z-b}\right) = const.$$

is an arc of a certain circle passing through the fixed points a and b,

we deduce that ab is a chord of a circle centred at infinity. In effect, (z-a) = (z-b) = ∞. At infinity, segments (z-a) and (z-b) would appear to be a single diameter and the angle between them approaches 0, which implies Im(W) approaches 0, but at our proximate viewer's location, they appear to be perpendicular to C. Since chord ab has finite length, it is indistinguishable from a segment of the circle itself (Figure 2). As z approaches ab from either end, (z-a) and (z-b) align with ab, so Im(W) = const. = 0. Then the locus of z is the infinite circle C through a and b, minus the finite segment from a to b.

If z lies between a and b on C (possible because in the limit, the chord ab becomes a segment of C), then arg(z-a) - arg(z-b) = π, which implies that

$$\mathrm{Im}\left(\frac{z-a}{z-b}\right) = 0$$

as stipulated in the problem. Hence, ab is also a locus of z, so the entire infinite circle passing through (1+i) and -(1+i) is a locus of z. As in the first part, point b cannot be a locus of z unless C is rotated by a infinitesimal angle about its centre.