Needham, Chapter 1, Exercise 1, p. 45.
G. Palmer, December 18, 2019, 9:45 am, PST

Out[743]=

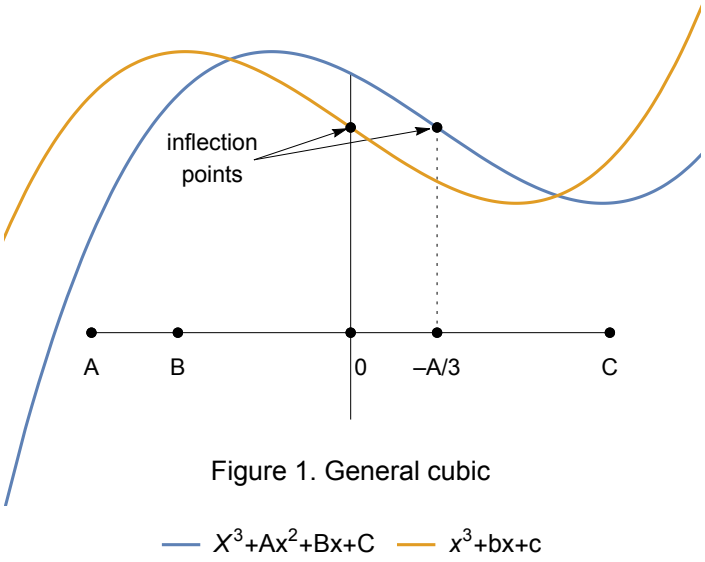


Figure 1. General cubic

1 The roots of a general cubic equation in X may be viewed (in the X Y-plane) as the intersections of the X-axis with the graph of a cubic of the form,

$$Y = X^3 + A X^2 + B X + C$$

(i) Show that the point of inflection of the graph occurs at $X = -\frac{A}{3}$.

The inflection point is where the 2nd derivative equals 0.

$$y' = 3 X^2 + 2AX + B$$

$$y'' = 6X + 2A$$

$$6X + 2A = 0$$

$$X = -\frac{A}{3}$$

(ii) Deduce (geometrically) that the substitution $(x - \frac{A}{3})$ *will reduce the above equation to the form* $Y = x^3 + bx + c$.

Plot the x-value of the inflection point for the substitution (orange line):

$$Y'' = 0 = 6(x - \frac{A}{3}) + 2\,A$$

$$6x - 2A + 2A = 0 \quad \Longrightarrow x = 0$$

$$Y = \left(-\frac{A}{3}\right)^3 + \; A\left(-\frac{A}{3}\right)^2 + B(-\frac{A}{3}) + C$$

$$= -\frac{A^3}{27} + \; A\;\frac{A^2}{9} - \frac{A\,B}{3} + C$$

$$= -\frac{A^3}{27} + \frac{3\,A^3}{27} - \frac{A\,B}{3} + C$$

$$= \frac{2\,A^3}{27} - \frac{A\,B}{3} + C$$

Figure 1 shows that inflection point of the second plot with X = (x - A/3) is on the y-axis and that the graph has symmetry about the inflection point with a half-rotation, but not about either of the axes. Hence this general cubic is an odd function, which can be written without an even power of x.

(iii) Verify this by calculation.

$$Y = (x - \frac{A}{3})^3 + A\left(x - \frac{A}{3}\right)^2 + BX + C$$

$$= -\frac{A^3}{27} + \frac{A^2\,x}{3} - A\,x^2 + x^3 + \frac{A^3}{9} - \frac{2\,A^2\,x}{3} + A\,x^2 + B\,(x - \frac{A}{3}) + C$$

$$= x^3 - \left(A^2 - B\right)x + \frac{2\,A^3}{27} - \frac{A\,B}{3} + C$$

Then if we let $b = \left(A^2 - B\right)$ and $c = \frac{2\,A^3}{27} - \frac{A\,B}{3} + C$ the equation has the format $Y = x^3 + bx + c$.