Needham, *Visual Differential Geometry and Forms*, Chapter 2, Notes and Exercises
G. Palmer, December 22, 9:30 am, PDT

P. 20. *With the assistance of the formula for the polar cap (see Ex. 10, p. 85) it is not hard to verify that, in fact,*

$$\mathcal{K} \asymp \frac{12}{\pi}\left[\frac{\pi r^2 - \mathrm{A}(r)}{r^4}\right].$$

From Ex. 10, p. 85, *the area $\mathcal{A}$ of the polar cap $0 \leq \phi \leq \Phi$* is

$$\mathcal{A} = 2\,\pi R^2(1 - \cos\Phi\,).$$

I could not find a definition for "polar cap", so I am assuming it is synonymous with the term "spherical cap," which is defined in Wikipedia as follows: *In geometry, a spherical cap or spherical dome is a portion of a sphere or of a ball cut off by a plane* <Spherical_cap>*.* Then l.c. ϕ is a variable that can take any value on the sphere, and u.c. Φ refers to the value of ϕ on the border of the polar cap.

Write a general expression for the difference between the areas of an euclidean disc $\mathcal{A}_E$ and a polar cap $\mathcal{A}_S$:

$$\mathcal{A}_E - \mathcal{A}_S = \pi r^2 - 2\,\pi \mathrm{R}^2(1 - \cos\Phi)$$

Use the power series for cosine to find an expression for $(1 - \cos\Phi)$, where $\Phi = \frac{r}{R}$.

$$\cos\phi = 1 - \frac{1}{2!}\,\Phi^2 + \frac{1}{4!}\,\Phi^4 - \ldots$$

$$(1 - \cos\phi) = \frac{1}{2!}\,\Phi^2 - \frac{1}{4!}\,\Phi^4 + \ldots \asymp \left(\frac{12}{12} \times \frac{1}{2}\,\Phi^2\right) - \frac{1}{24}\,\Phi^4$$

$$= \frac{1}{24} \times 12\left(\frac{r}{R}\right)^2 - \frac{1}{24}\left(\frac{r}{R}\right)^4$$

$$= \left((12\,r^2)/(24\,R^2)\right)\left(R^2/R^2\right) - r^4/(24\,R^4) = \left(12\,R^2\,r^2 - r^4\right)/\left(24\,R^4\right) \qquad (1)$$

[Mathematica 12 reverted to a linear format.]

It follows that as r shrinks to 0,

$$(\mathcal{A}_E - \mathcal{A}_S) \asymp \pi r^2 - 2\,\pi\,R^2\left(\left(12\,R^2\,r^2 - r^4\right)/\left(24\,R^4\right)\right) \qquad (2)$$

Since $\mathcal{K} = 1/R^2$, $\mathcal{K}$ is the excess in area on the sphere divided by the coefficient of $1/R^2$. The calculation of [2.4] worked out nicely because the excess circumference was reduced to a compact quotient with r^3

in the numerator and R^2 in the denominator. We attempt to reduce the RHS of (2) to such a quotient.

$$(\mathcal{A}_E - \mathcal{A}_S) \asymp \left(24\,R^4\,\pi r^2\right)/\left(24\,R^4\right) - 2\,\pi\,R^2\left(\left(12\,R^2\,r^2 - r^4\right)/\left(24\,R^4\right)\right)$$

$$= \left(24\,R^4\,\pi r^2 - 24\,\pi R^4\,r^2 + 2\,\pi R^2\,r^4\right)/\left(24\,R^4\right) = \left(2\,\pi R^2\,r^4\right)/\left(24\,R^4\right) = \pi r^4/\left(12\,R^2\right)$$

Now divide through by the coefficient of $1/R^2$ to obtain $\mathcal{K}$.

$$\mathcal{K} = 1/R^2 \asymp -\left((12\,(\mathcal{A}_E - \mathcal{A}_S))/\pi r^4\right) \asymp \frac{12}{\pi}\left[\left(\pi r^2 - \mathcal{A}(r)\right)/r^4\right]$$

We know that the curvature has dimension $1/(\text{length})^2$ and area in the numerator is $(\text{length})^2$, so r takes the 4th power in the denominator.