Needham, VDF, Prologue, pp. xviii, xix.
G. Palmer, November 12, 2021, 8:45 pm PST

It follows [exercise] from the theorems on limits that ultimate equality is an equivalence relation.

Equivalence relation: a relation that is reflexive, symmetric, and transitive (Gellert, et al. The VNR Concise Encyclopedia of Mathematics, 1977, p. 323).

Presumably, the three criteria apply as follows:

For A, B, C in set S,

reflexive: $A \asymp A$

symmetric: $A \asymp B \Longrightarrow B \asymp A$

transitive: $A \asymp B \asymp C \Longrightarrow A \asymp C$

Theorems on limits (Gellert, et al, p. 398; Gellert refers to “calculations” rather than “theorems”, but I think they are the same as what others are calling the limit theorems, laws, or properties. See, for example, https://people.math.umass.edu/~gunnells/teaching/Sample_Lecture_Notes.pdf).

Where $\lim_{x \to l} f = F$ and $\lim_{x \to l}(g) = G$,

1) $\lim_{x \to l}(f + g) = F + G$
2) $\lim_{x \to l}(\mathrm{fg}) = \mathrm{FG}$
3) $\lim_{x \to l}(f / g) = F / G$

Assume non-zero quotients, and show that “ultimate equality” is reflexive, symmetric, and transitive.

It is given that

$$A \asymp B \Longleftrightarrow \lim_{\epsilon \to 0}\left(\frac{A}{B}\right) = 1 \qquad (1)\ (\text{p. xviii})$$

Reflexive

A can be substituted for B in (1) because both are members of S.

$$A \asymp A \Longleftrightarrow \lim_{\epsilon \to 0}\left(\frac{A}{A}\right) = 1$$

so $\asymp$ is reflexive.

Symmetric

A and B can be substituted for one another in (1) because both are members of S.

$$B \asymp A \iff \lim_{\epsilon\to 0}\left(\frac{B}{A}\right) = 1 = \lim_{\epsilon\to 0}\left(\frac{A}{B}\right) \iff A \asymp B$$

Then B ≍ A is symmetric to A ≍ B, and ≍ is shown to be symmetric.

Transitive

If $A \asymp B \iff \lim_{\epsilon\to 0}\left(\frac{A}{B}\right) = 1$,
and $B \asymp C \iff \lim_{\epsilon\to 0}\left(\frac{B}{C}\right) = 1$,
$A \asymp C \iff \lim_{\epsilon\to 0}\left(\frac{A}{C}\right) = \lim_{\epsilon\to 0}\left(\frac{A}{B}\,\frac{B}{C}\right) = \lim_{\epsilon\to 0}\left(\frac{A}{B}\right)\lim_{\epsilon\to 0}\left(\frac{B}{C}\right) = 1\cdot 1 = 1$
(by limit theorem 3)

“Ultimate equality” (≍) meets the three criteria for an equivalence relation. It appears that only the third limit theorem is needed to establish that ≍ is an equivalence relation.